MATH 1910 EXAM III

100 points

NAME:____________________________

Write down the antiderivative family for each function below. These are worth five points each.

(1) $f(x) = \cos(x)$ $\qquad \int f(x)dx = \sin(x) + C$

(2) $g(y) = \ln(y)$ $\qquad \int g(y)dy = y\ln(y) - y + C$

(3) $h(z) = \frac{1}{z}$ $\qquad \int h(z)dz = \ln|z| + C$

(4) $k(a) = a^3$ $\qquad \int k(a)da = \frac{a^4}{4} + C$

(5) $j(b) = b^{1/2}$ $\qquad \int j(b)db = \frac{2b^{3/2}}{3} + C$

10 pts **6.** Are either of the limits below an indeterminate form? If so, indicate whether the limit is of the form 0/0 or ∞/∞. If it is not an indeterminate form, explain why not.

(a) $\lim_{t\to 3} \frac{\ln(t-3)}{2t}$ This limit is not indeterminate; while $\lim_{t\longrightarrow 3}\ln(t-3) = -\infty$, we have $\lim_{t\longrightarrow 3}(2t) = 6$.

(b) $\lim_{\theta\to 0} \frac{\tan(\theta)-1}{2-\cos(\theta)}$ This limit is not indeterminate; we have $\lim_{\theta\longrightarrow 0}(\tan(\theta)-1) = -1$.

10 pts **7.** The limit below is indeterminate. Indicate which indeterminate form the limit has, then use L'Hôpital's Rule to evaluate the limit. You must show your work for full credit.

$$\lim_{x\to 1} \frac{\sin(\pi x)}{x^2-1}$$

Since $\lim_{x\longrightarrow 1}\sin(\pi x) = 0 = \lim_{x\longrightarrow 1}(x^2-1)$, this limit has indeterminate form 0/0. Now,

$$\lim_{x\to 1} \frac{\sin(\pi x)}{x^2-1} \underbrace{=}_{\text{LHR}} \lim_{x\longrightarrow 1} \frac{\pi\cos(\pi x)}{2x} = -\frac{\pi}{2}$$

15 pts **8.** Consider the function $f(x) = \dfrac{x-1}{x^2}$.

Part (a) What is the derivative for f? You must show your work for full credit.

$$\begin{aligned} f'(x) &= \left(\frac{1}{x^4}\right)\left(x^2\frac{d}{dx}[x-1] - (x-1)\frac{d}{dx}\left[x^2\right]\right) \\ &= \frac{x^2(1)-(x-1)(2x)}{x^4} \\ &= \frac{2x-x^2}{x^4} \\ &= \frac{2-x}{x^3} \end{aligned}$$

Part (b) What are the critical numbers for f? You must show your work for full credit. (Do not classify the crtitical numbers.)

The derivative will be undefined when $x = 0$ and will have output 0 when $2 - x = 0$; that is, when $x = 2$. These are the only critical numbers for f.

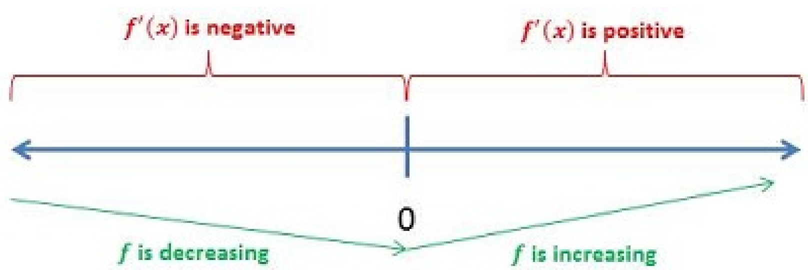
10 pts **11.** A function f and its first derivative are given below.

$$f(x) = \ln\left(x^4 + x^2 + 1\right) \qquad f'(x) = \frac{4x^3+2x}{x^4+x^2+1}$$

This function has a single critical number at $x = 0$. (You do not have to show this.) Use the First Derivative Test to classify this critical number as a value where f has either a relative maximum output or a relative minimum output. You must show your work for full credit.

- Use $x = -1$ as a test number and observe that $f'(-1) = \frac{4(-1)^3+2(-1)}{(-1)^4+(-1)^2+1} < 0$. Hence, the graph of f is falling to the left of $x = 0$.

- Use $x =$ as a test number and observe that $f'(1) = \frac{4(1)^3+2(1)}{(1)^4+(1)^2+1} > 0$. Hence, the graph of f is rising to the left of $x = 0$.



Hence, f will have a relative minimum output at $x = 0$. As a matter of fact, f will have an absolute minimum output at $x = 0$.

20 pts **12.** Suppose we want to find numbers m and n, both greater than or equal to 0, whose sum is fixed at 300 and whose product P is *as small as possible.*

(a) What restrictions do we have?

We are told that $m + n = 300$, and we are told that $0 \leq m$ and $0 \leq n$.

(b) Written in terms of m, what is the optimization function for this problem?

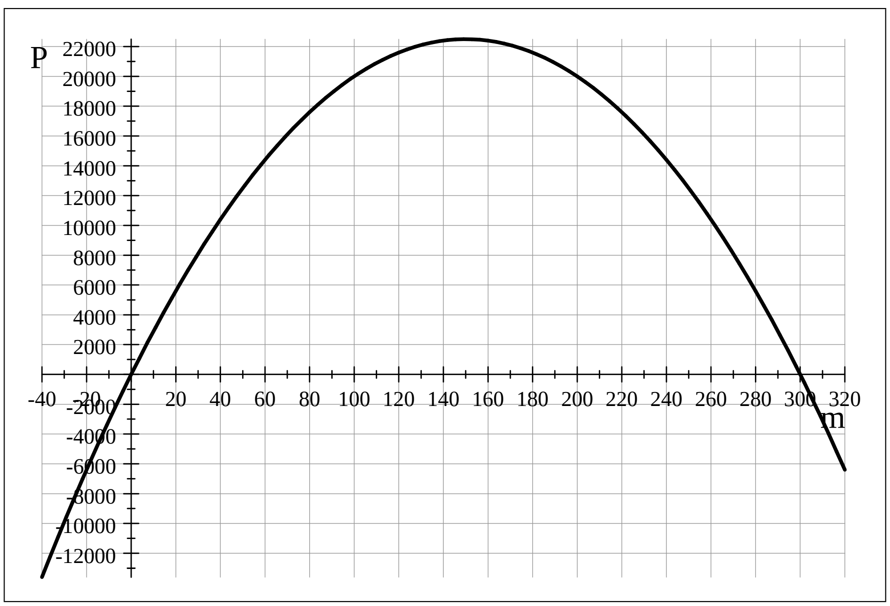
We are told to optimize the product of the two numbers. Therefore, the optimization formula is $P = mn$. We know that $y = 300 - m$; therefore, the optimization function is

$$P = f(m) = m(300 - m)$$

(c) What is the relevant domain for this function?

We know that $0 \leq m$. Since it must also be the case that $0 \leq y$, we also know that $m \leq 300$. The relevant domain is therefore $0 \leq m \leq 300$.

(d) The diagram below shows a graph of the optimization function on an interval that is larger than the relevant domain. Using this graph and the relevant domain, which value or values of m solve the problem?



The smallest output for the function f occurs at the endpoints of the relevant domain; in particular, when $m = 0$ and $m = 300$. In both cases, the minimum output is 0.

10 pts **14.** Find the antiderivative family for the function $f(x) = 3\sin(x) + \frac{2}{x} - 5$. You must show your steps for full credit.

$$\begin{aligned}\int \left(3\sin(x) + \frac{2}{x} - 5\right) dx &= 3\int \sin(x)dx + 2\int \frac{1}{x}dx - \int 5dx \\ &= -3\cos(x) + 2\ln|x| - 5x + C\end{aligned}$$

BONUS:

10 pts A rectangle must have a fixed area of 1000 square inches. If we want to construct the rectangle so that that its perimeter is as small as possible, what is the optimization function, written in terms of width only?

Let W and L represent the width and length, respectively, for the rectangle, and assume both are measured in inches. Let P represent the perimeter of the rectangle, also measured in inches. We are told to optimize the perimeter, so the optimization formula will be

$$P = 2W + 2L$$

We are also told that $1000 = WL$, therefore, we can rewrite the optimization formula as a function of W alone. Doing so, we obtain

$$P = f(W) = 2W + \frac{2000}{W}$$