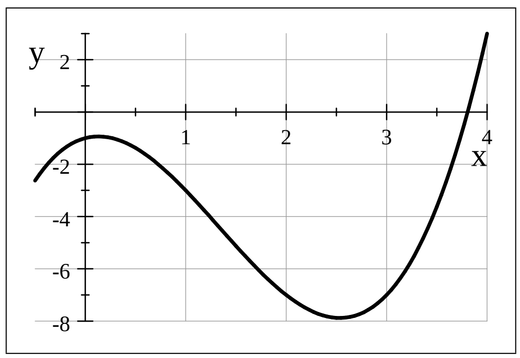
MATH 1910 PRACTICE EXAM III

1. ____**D**____ Suppose that a function f is increasing and continuous on the half-open interval $-2 < x \leq 1$. Which of the following statements must be true?

   (a) The function f has no absolute extrema.
   (b) The function f has an absolute minimum at $x = 1$ and an absolute maximum at $x = -2$.
   (c) The function f has an absolute minimum at $x = -2$ and an absolute maximum at $x = 1$.
   (d) The function f has an absolute maximum at $x = 1$ and no absolute minimum.
   (e) The function f has an absolute minimum at $x = -2$ and no absolute maximum.

   Justification. Since the graph of the function f is increasing (viewed from left to right) and has no tears, its largest output value would have to occur at $x = 1$. The smallest output value would have to occur at $x = -2$; however, this endpoint is not included in the specified set.

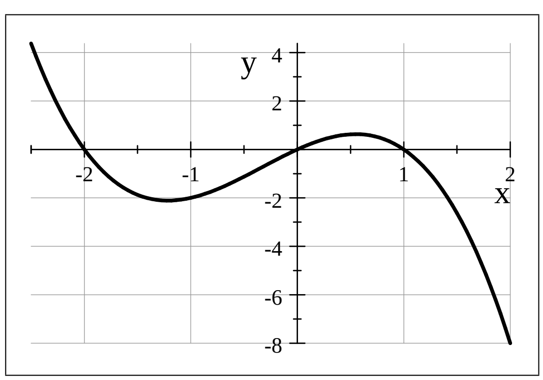
2. ____**C**____ Suppose a function f is defined on the segment $-0.5 < x < 4$ by the graph shown below. Which of the following statements must be true?



   (a) The function f has no absolute extrema.
   (b) The function f has an absolute maximum and an absolute minimum.
   (c) The function f has an absolute minimum but no absolute maximum.
   (d) The function f has an absolute maximum but no absolute minimum.
   (e) The function f has two absolute maxima but no absolute minima.

   Justification. The largest output value for the function occurs when $x = 4$; however, this endpoint is not included in the specified set. The absolute minimum output for the function occurs at $x \approx 2.6$.

The diagram below shows the *derivative* graph for a function f. Use this graph to answer Problems 3 and 4.



3. ____**A**____ Based on the *derivative* graph shown above, we see that f has critical numbers at

(a) $x = -2$, $x = 0$, and $x = 1$. (b) $x = -1.25$ and $x = .6$.

(c) only $x = .6$. (d) only $x = -1.25$.

(e) $x = -2.5$ and $x = 2$.

4. ____**A**____ Based on the *derivative* graph shown above, we see that f has relative maximum outputs

(a) at $x = -2$ and $x = 1$. (b) only at $x = 0$.

(c) only at $x = .6$. (d) only at $x = -1.25$.

(e) at $x = -2.5$ and $x = 2$.

Justification. As we move across the input value $x = -2$ from left to right, the output sign of the derivative function changes from positive to negative. Consequently, the First Derivative Test tells us that the function f will have a relative maximum output at $x = 2$. The same holds for $x = 1$.

Problems 5 and 6 refer to the function and its derivatives shown below.

$$f(x) = \ln\left(2x^2 - 2x + 1\right) \qquad f'(x) = \frac{2(2x-1)}{2x^2 - 2x + 1}$$

5. ____**A**____ The function f will have critical numbers

(a) only at $x = 1/2$. (b) at $x = 0$ and $x = 1$.

(c) at $x = \frac{1}{2}\left(1 \pm \sqrt{3}\right)$. (d) at $x = 1/2$ and $x = \frac{1}{2}\left(1 \pm \sqrt{3}\right)$.

(e) at no value of x.

Justification. Setting the numerator of f' equal to 0, we have $2x - 1 = 0$, and this tells us that $f'(x) = 0$ when $x = 1/2$. Setting the denominator of the function f' equal to 0 produces no real solutions.

6. ____**C**____ On the interval $[1, 3]$, the function f will have its absolute minimum output at

(a) $x = 3$ (b) $x = \frac{1}{2}\left(1 + \sqrt{3}\right)$

(c) $x = 1$ (d) $x = 1/2$

(e) $x = 2$

Justification. The critical number $x = 1/2$ is not included in the specified set, so we only need to evaluate the function f at the endpoints $x = 1$ and $x = 3$. The smallest output occurs at the input value $x = 1$.

7. Determine the formula for $f'(x)$ if $y = f(x) = x\ln(\sin(x))$.

$$\frac{df}{dx} = \frac{d}{dx}[x]\ln(\sin(x)) + x\frac{d}{dx}[\sin(x)]\left.\frac{d}{du}[\ln(u)]\right|_{u=\sin(x)} = x\ln(\sin(x)) + \frac{\cos(x)}{\sin(x)}$$

8. Differentiate the function $y = f(x) = \arcsin(\log_2(x))$ with respect to the variable x.

$$\frac{df}{dx} = \frac{d}{dx}\left[\log_2(x)\right] \frac{d}{du}\left[\arcsin(u)\right]\Big|_{u=\log_2(x)} = \left(\frac{1}{x\ln(2)}\right)\frac{1}{\sqrt{1-\left(\log_2(x)\right)^2}}$$

9. For what values of x will the tangent line to the function $y = f(x) = \ln(1-x^2)$ have slope $m = 2$?

$$\frac{df}{dx} = \frac{d}{dx}\left[1-x^2\right] \frac{d}{du}\left[\ln(u)\right]\Big|_{u=1-x^2} = -\frac{2x}{1-x^2}$$

$$\begin{aligned} -\frac{2x}{1-x^2} = 2 &\implies 2x = 2\left(x^2-1\right) \\ &\implies 2x^2 - 2x - 2 = 0 \\ &\implies x^2 - x - 1 = 0 \\ &\implies x = \frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{aligned}$$

10. For what values of x will the tangent line to the function $y = f(x) = \arctan(x^3 - x + 2)$ be horizontal?

$$\begin{aligned} \frac{df}{dx} &= \frac{d}{dx}\left[x^3 - x + 2\right] \frac{d}{du}\left[\arctan(u)\right]\Big|_{u=x^3-x+2} \\ &= (3x^2-1)\left(\frac{1}{1+\left(x^3-x+2\right)^2}\right) \end{aligned}$$

Setting $f'(x) = 0$ gives us the equation $3x^2 - 1 = 0$. The solutions to this equation are $x = \pm\frac{1}{\sqrt{3}}$.

11. Find a formula for $\frac{dx}{dt}$ if $t = x^2 + \tan(x)$.

$$\begin{aligned} t = x^2 + \tan(x) &\implies \frac{d}{dt}\left[t\right] = \frac{d}{dt}\left[x^2 + \tan(x)\right] \\ &\implies 1 = \left(2x + \sec^2(x)\right)\frac{dx}{dt} \\ &\implies \frac{1}{2x+\sec^2(x)} = \frac{dx}{dt} \end{aligned}$$

12. Find a formula for $\frac{dy}{dx}$ if $y^3 - y = x^3$.

$$\begin{aligned} y^3 - y = x^3 &\implies \frac{d}{dx}\left[y^3 - y\right] = \frac{d}{dx}\left[x^3\right] \\ &\implies \left(3y^2-1\right)\frac{dy}{dx} = 3x^2 \\ &\implies \frac{dy}{dx} = \frac{3x^2}{3y^2-1} \end{aligned}$$

13. What is the slope of the tangent line to the graph of $y^3 - y = x^2$ at the point $(\sqrt{6}, 2)$?

$$\begin{aligned} y^3 - y = x^2 \quad &\Longrightarrow \quad \frac{d}{dx}\left[y^3 - y\right] = \frac{d}{dx}\left[x^2\right] \\ &\Longrightarrow \quad \left(3y^2 - 1\right)\frac{dy}{dx} = 2x \\ &\Longrightarrow \quad \frac{dy}{dx} = \frac{2x}{3y^2 - 1} \end{aligned}$$

$$\left.\frac{dy}{dx}\right|_{(\sqrt{6},2)} = \frac{2\sqrt{6}}{11}$$

14. What is the slope of the tangent line to the graph of $x\cos(y) = y$ at the point $(0, 0)$?

$$\begin{aligned} x\cos(y) = y \quad &\Longrightarrow \quad \frac{d}{dx}\left[x\cos(y)\right] = \frac{d}{dx}\left[y\right] \\ &\Longrightarrow \quad \cos(y) - x\sin(y)\frac{dy}{dx} = \frac{dy}{dx} \\ &\Longrightarrow \quad \frac{dy}{dx} = \frac{\cos(y)}{1 + x\sin(y)} \end{aligned}$$

$$\left.\frac{dy}{dx}\right|_{(0,0)} = 1$$

15. Find a formula for $\dfrac{dv}{du}$ if $v\sin(u) = u\sin(v)$

$$\begin{aligned} v\sin(u) = u\sin(v) \quad &\Longrightarrow \quad \frac{d}{du}\left[v\sin(u)\right] = \frac{d}{du}\left[u\sin(v)\right] \\ &\Longrightarrow \quad \sin(u)\frac{dv}{du} + v\cos(u) = \sin(v) + u\cos(v)\frac{dv}{du} \\ &\Longrightarrow \quad \frac{dv}{du} = \frac{\sin(v) - v\cos(u)}{\sin(u) - u\cos(v)} \end{aligned}$$

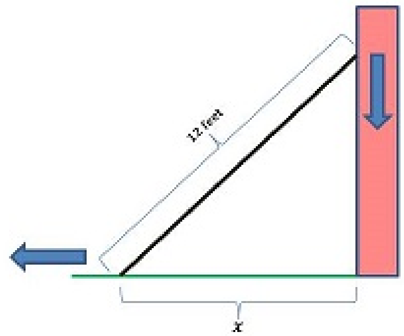
16. Differentiate the formula $V = \pi r^2 h$ with respect to the variable t.

$$V = \pi r^2 h \Longrightarrow \frac{dV}{dt} = \pi\frac{d}{dt}\left[r^2 h\right] \Longrightarrow \frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

17. Differentiate the formula $x^2 + y^2 = h^2$ with respect to the variable t.

$$x^2 + y^2 = h^2 \Longrightarrow \frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[h^2\right] \Longrightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = h\frac{dh}{dt}$$

18. Suppose a 12-foot ladder is leaning against a wall that is perpendicular to the ground. Emily pulls the base of the ladder away from the wall. Let t represent the number of seconds passed since Emily began pulling the base of the ladder, and let x represent the distance in feet between the base of the ladder and the base of the wall. When $x = 6$ feet, suppose we know that the instantaneous rate of change in the values of x with respect to the values of t is 5.5 feet per second. What is the corresponding instantaneous rate of change in the vertical distance between the top of the ladder and the ground (measured in feet) with respect to the values of t?



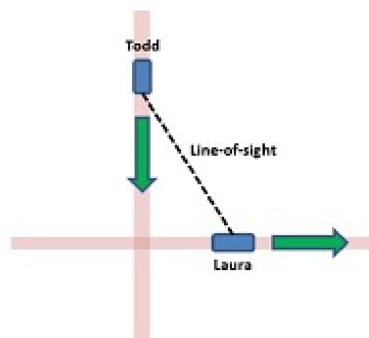
Solution. First, let y denote the vertical distance between the top of the ladder and the ground (measured in feet). The Pythagorean Theorem and implicit differentiation tell us

$$144 = x^2 + y^2 \qquad 0 = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Now, when $x = 6$ feet, we know that $y = \sqrt{144 - 36} \approx 10.4$ feet. Therefore, we know

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \Longrightarrow \frac{dy}{dt} \approx -\left(\frac{6\text{ feet}}{10.4\text{ feet}}\right)\left(5.5\frac{\text{feet}}{\text{second}}\right) \approx -3.2\frac{\text{feet}}{\text{second}}$$

19. Suppose a straight north-south road intersects a straight east-west road. When Todd is 0.75 miles from the intersection on the north-south road, his distance from the intersection (measured in miles) is decreasing at 20 miles per hour. At the same moment, Laura is 0.25 miles from the intersection on the east-west road, and her distance from the intersection (measured in miles) is increasing at 15 miles per hour. What is the instantaneous rate of change of the line-of-sight distance between Laura and Todd with respect to time?



Solution. First, let y denote thedistance betweenTodd and the intersection (measured in miles) and let x denote the distance between Laura and the intersection (measured in miles). Let h represent the line-of-sight distance between Todd and Laura (measured in miles), and let t represent the number of hours passed (relative to a starting time prior to Todd's and Laura's distances from the intersection being 0.75 and 0.25 miles, respectively). The Pythagorean Theorem and implicit differentiation tell us

$$h^2 = x^2 + y^2 \qquad h\frac{dh}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Now, when $x = 0.25$ miles and $y = 0.75$ miles, we know that $h = \sqrt{0.25^2 + 0.75^2} \approx 0.79$ miles. Therefore, we know

$$\begin{aligned}\frac{dh}{dt} &= \frac{1}{h}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)\\ \Longrightarrow \frac{dh}{dt} &\approx -\left(\frac{1}{0.79\text{ mile}}\right)\left[(0.75\text{ mile})\left(-20\frac{\text{miles}}{\text{hour}}\right) + (0.25\text{ mile})\left(15\frac{\text{miles}}{\text{hour}}\right)\right] \approx 14.24\frac{\text{miles}}{\text{hour}}\end{aligned}$$

20. The volume of a cylinder (measured in cubic inches) is given by the formula $V = \pi r^2 h$, where r and h represent the radius and height, respectively, for the cylinder (both measured in inches). Suppose that a cylindrical piece of ice is melting in such a way that the length of its radius is always twice its height. Let t represent the number of minutes passed since the cylinder began to melt, and suppose we know that when the cylinder is three inches high, the instantaneous rate of change in the radius with respect to time is -3.5 inches per minute. What is the corresponding instantaneous rate of change in the volume of the cylinder with respect to time?

Solution. First, note that we are told that $r = 2h$. Consequently, we know

$$V = \pi r^2 h \Longrightarrow V = 4\pi h^3$$

$$\frac{dr}{dt} = 2\frac{dh}{dt} \qquad \frac{dV}{dt} = 12\pi h^2 \frac{dh}{dt}$$

Now, at the instant in time when $h = 3$ inches, we know

$$r = 6 \text{ inches} \qquad \frac{dr}{dt} = -3.5\frac{\text{inches}}{\text{minute}} \qquad \frac{dh}{dt} = -1.75\frac{\text{inches}}{\text{minute}}$$

Therefore, we also know

$$\frac{dV}{dt} = 12\pi h^2 \frac{dh}{dt} \Longrightarrow \frac{dV}{dt} = 12\pi\,(3 \text{ inches})^2 \left(-1.75\frac{\text{inches}}{\text{minute}}\right) \approx -593.8\frac{\text{cubic inches}}{\text{minute}}$$