MATH 1910 PRACTICE EXAM IV

1. **B** One antiderivative for $f(x) = e^x$ is the function

   (a) $F(x) = \frac{1}{2}e^{x^2}$ (b) $F(x) = e^x - \sqrt{2}$

   (c) $F(x) = \frac{1}{2}e^{2x}$ (d) $F(x) = \ln(x) + \pi$

   (e) $F(x) = 3e^x$

   Justification. We must identify a function whose derivative function is the function f. Of all the choices, only

$$\frac{d}{dx}\left[e^x - \sqrt{2}\right] = e^x$$

2. **B** In order to compute the antiderivative family for $f(x) = \frac{\cos(1/x)}{x^2}$ we need the substitution

   (a) $u = \cos(x)$ (b) $u = \frac{1}{x}$

   (c) $u = \frac{1}{x^2}$ (d) $u = x^2$

   (e) $u = x$

   Justification. We must identify a function-derivative pair. By letting $u = 1/x$, we see that

$$\frac{\cos(1/x)}{x^2} = \cos(1/x)\left(\frac{1}{x^2}\right) = \cos(u)\left(-\frac{du}{dx}\right) = -\cos(u)\frac{du}{dx}$$

3. **A** The function $F(x) = x\sin(x) + 10$ is an antiderivative for which of the following functions?

   (a) $f(x) = x\cos(x) + \sin(x)$ (b) $f(x) = \frac{x^2}{2}\sin(x) - \cos(x)$

   (c) $f(x) = -\frac{x^2}{2}\cos(x)$ (d) $f(x) = x\cos(x)$

   (e) $f(x) = \frac{x^2}{2}\cos(x)$

   Justification. We must identify the function f with the property that $F'(x) = f(x)$. Observe

$$F'(x) = \frac{d}{dx}\left[x\sin(x) + 10\right] = x\frac{d}{dx}\left[\sin(x)\right] + \sin(x)\frac{d}{dx}\left[x\right] + \frac{d}{dx}\left[10\right] = x\cos(x) + \sin(x)$$

4. **C** The second derivative for the function $y = f(x) = x\ln(x) - x$ is the function

   (a) $f''(x) = 0$ (b) $f''(x) = \ln(x)$

   (c) $f''(x) = \frac{1}{x}$ (d) $f''(x) = -\frac{1}{x^2}$

   (e) $f''(x) = 1 + \frac{1}{x\ln(x)}$

   Justification. We know that $f'(x) = \ln(x)$; hence, $f''(x) = 1/x$.

5. $\underline{\quad\textbf{E}\quad}$ By making an appropriate substitution, we know that $\displaystyle\int \frac{x^2}{(1+x^3)^4}dx$

(a) is equal to $3\displaystyle\int \frac{u}{(1+u)}du$ (b) is equal to $\displaystyle\int x^2\left(\frac{1}{u^4}\right)du$

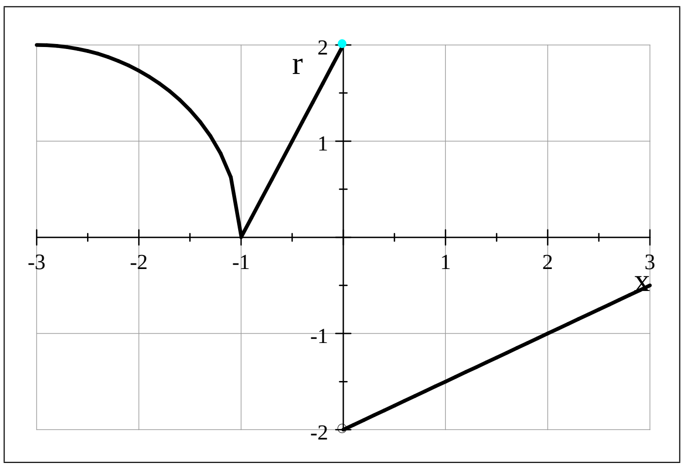
(c) is equal to $\displaystyle\frac{1}{2}\int \frac{u}{1+u^6}du$ (d) is equal to $\displaystyle\int u^{-4}du$

(e) is equal to $\displaystyle\frac{1}{3}\int u^{-4}du$

Justification. The appropriate substitution is $u = 1 + x^3$, since this is the only choice for u that produces a function-derivative pair. With this choice, we have

$$\int \frac{x^2}{(1+x^3)^4}dx = \int \frac{1}{(1+x^3)^4}\left(x^2\right)dx = \int \frac{1}{u^4}\left(\frac{1}{3}\cdot\frac{du}{dx}\right)dx = \frac{1}{3}\int u^{-4}du$$

Problems 6-7 refer to the graph of a function $r = f(x)$ below. Suppose that $y = F(x)$ is an antiderivative for the function f.



6. $\underline{\quad\textbf{E}\quad}$ What are the critical numbers for the function F in the viewing window shown?

(a) Critical numbers are $x = -3$ and $x = -1$ (b) Critical number is $x = -3$

(c) Critical number is $x = 0$ (d) Critical number is $x = -1$

(e) Critical numbers are $x = -1$ and $x = 0$

Justification. Since F is an antiderivative for the function f, we know that the graph above shows the derivative function for the function F. Therefore, the critical numbers for F will be those input values where the function $f(x) = 0$ or $f(x)$ is undefined.

7. $\underline{\quad\textbf{A}\quad}$ In the viewing window shown, at what value of x does the function F have a local minimum output?

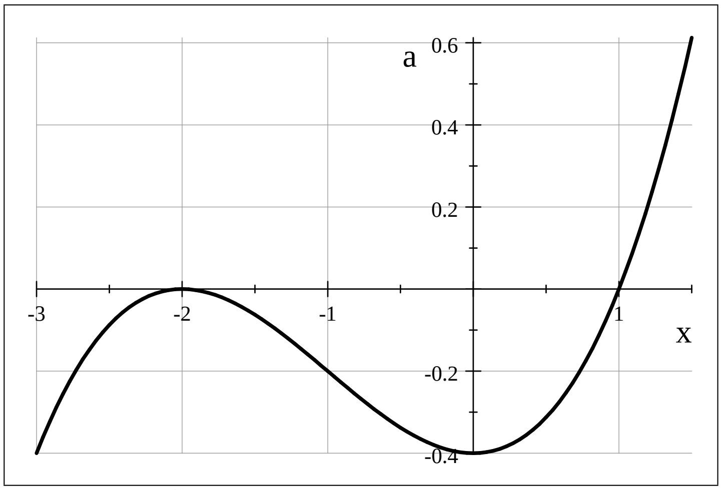
(a) at no value of x (b) at $x = -3$

(c) at $x = 0$ (d) at $x = -1$

(e) at $x = 3$

Justification. The sign of the output values from the function f does not change as we move across the input value $x = -1$. The sign of the output values for the function f change from positive to negative as we move across the input value $x = 0$. Hence, the function F has a local maximum output at $x = 0$.

Problems 8 and 9 refer to the graph below. This graph shows the *second derivative* function for a function $y = f(x)$.



Second Derivative Function for f

8. ____**D**________ Based on the graph shown above, where does the function f have inflection points?

   (a) $x = -1$ only (b) $x = -2$ and $x = 1$

   (c) $x = 0$ only (d) $x = 1$ only

   (e) no value of x

   Justification. Based on the graph, we see that the derivative function f' has two critical numbers, namely $x = -2$ and $x = 1$. The sign of the output values from the function f'' change as we move across the critical number $x = 1$; hence, we may conclude that the original function f has an inflection point at this input value.

9. ____**D**________ Based on the graph shown above, on what input intervals is the graph of the function f concave down?

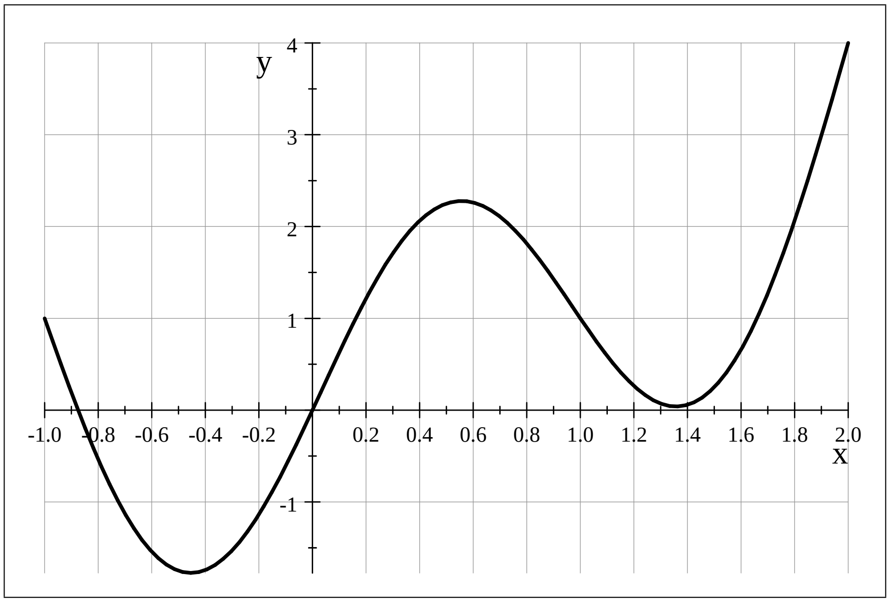
   (a) $-\infty < x < -1$ (b) $-2 < x < 1$

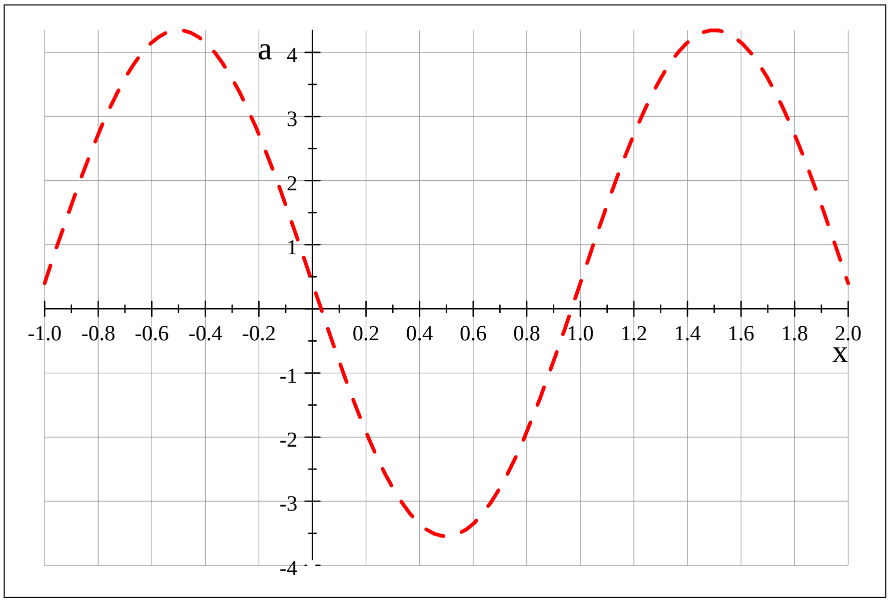
   (c) $1 < x < +\infty$ (d) $-\infty < x < 1$

   (e) $-1 < x < 1$

   Justification. The original function f will be concave down when the output values of the second derivative function are negative.

10. The diagram below shows the graph of a function $y = f(x)$. On the grid provided, sketch the graph of the second derivative function for the function f.





11. If f is the function whose graph is shown in Problem 10, what are the critical numbers for its derivative function f'?

    Solution. Based on the graph, we know that the derivative function will have critical numbers at $x \approx 0.05$, and $x \approx 0.95$. There may also be critical numbers just to the left of $x = -1$ and just to the right of $x = 2$, but we don't have enough information to be sure.

12. Construct the second derivative function for the function $y = f(x) = \arctan(x)$.

$$f''(x) = -\frac{2x}{\left(1 + x^2\right)^2}$$

13. Construct the second derivative function for the function $y = f(x) = \ln(\sin(x))$.

$$f''(x) = -\csc^2(x)$$

14. Suppose that $y = f(x) = \ln(1 + x^2)$. What are the critical numbers for the function? Does the function f have any local maximum or minimum outputs?

Solution. First, observe that

$$f'(x) = \frac{2x}{1 + x^2}$$

Therefore, the function f has only one critical number, namely $x = 0$. Using test values $x = -1$ and $x = 1$, we see that $f'(-1) < 0$ while $f'(1) > 0$. Consequently, the First Derivative Test tells us that the function f has a local minimum output at $x = 0$.

15. Suppose that $y = f(x) = \ln(1 + x^2)$. What are the critical numbers for the derivative function $r = f'(x)$? Are any of these critical numbers inflection points for the function f?

Solution. First, observe that

$$f''(x) = \frac{2 - 2x^2}{(1 + x^2)^2}$$

Therefore, the derivative function f' has two critical numbers, namely $x = \pm 1$. Using test values $x = -2$, $x = 0$ and $x = 2$, we see that $f''(-2) < 0$, $f''(0) > 0$, and $f''(2) < 0$. Consequently, the First Derivative Test tells us that the derivative function f' has a local minimum output at $x = -1$ and a local maximum output at $x = 1$. Consequently, the original function f has an inflection point at both of these input values.

16. Suppose that the derivative function for a function $y = f(x)$ is given below. Based on this formula, at which input values does the function f have a local maximum and/or a local minimum output?

$$f'(x) = \frac{x - 2}{x^2}$$

Solution. First, notice that the function f has two critical numbers, namely $x = 0$ and $x = 2$. Consider the test values $x = -1$, $x = 1$, and $x = 3$. Observe that

$$f'(-1) < 0 \qquad f'(1) < 0 \qquad f'(3) > 0$$

Based on this information, the First Derivative Test tells us that the function f does not have a local maximum or minimum output at $x = 0$. On the other hand, the function f has a local minimum output at $x = 3$.

17. Suppose that the second derivative for a function $y = f(x)$ is given below. Based on this formula, at which input values does the *derivative* function $r = f'(x)$ have a local maximum and/or a local minimum output?

$$f''(x) = \frac{x^2 - 9}{x - 1}$$

Solution. First, notice that the derivative function has three critical numbers, namely $x = -3$, $x = 1$, and $x = 3$. Consider the test values $x = -4$, $x = 0$, $x = 2$, and $x = 4$. Observe that

$$f''(-4) < 0 \qquad f''(0) > 0 \qquad f''(2) < 0 \qquad f''(4) > 0$$

Based on this information, the First Derivative Test tells us that the derivative function has a local minimum output at $x = -3$ and at $x = 3$ and has a local maximum output at $x = 1$. The original function f will therefore have an inflection point at each of these input values.

18. Show that $F(x) = \sqrt{2} + e^x \sin(x)$ is one antiderivative for the function $f(x) = e^x(\sin(x) + \cos(x))$.

Solution. Apply the sum and the product rule to the function F.

19. Show that $F(x) = x \arcsin(x) + 8$ is one antiderivative for the function $f(x) = \dfrac{\sqrt{1-x^2}\arcsin(x) + x}{\sqrt{1-x^2}}$.

Solution. Apply the sum and the product rule to the function F, then rewrite the derivative as a single fraction.

20. Construct the antiderivative family for the function $y = f(x) = 3x + \ln(x)$.

$$\int (3x + \ln(x))\, dx = \frac{3}{2}x^2 + x\ln(x) - x + C$$

21. Construct the antiderivative family for the function $y = f(x) = \dfrac{4\sec^2(x) - x^{1/3}}{\pi}$.

$$\int \frac{4\sec^2(x) - x^{1/3}}{\pi} dx = \frac{1}{\pi}\left(4\tan(x) - \frac{3}{4}x^{4/3}\right) + C$$

22. Evaluate $\displaystyle\int x\cos(x^2+1)dx$.

Solution. Let $u = x^2 + 1$. Then we have

$$\begin{aligned}
\int x\cos(x^2+1)dx &= \int \cos(x^2+1)\,(x)\,dx \\
&= \int \cos(u)\left(\frac{1}{2}\cdot\frac{du}{dx}\right) dx \\
&= \frac{1}{2}\int \cos(u)du \\
&= \frac{1}{2}\sin(x^2+1) + C
\end{aligned}$$

23. Evaluate $\displaystyle\int \frac{\sin\left(x^{-1}\right)}{x^2} dx$.

Solution. Let $u = x^{-1}$. Then we have

$$\begin{aligned}
\int \frac{\sin\left(x^{-1}\right)}{x^2} dx &= \int \sin(x^{-1})\left(\frac{1}{x^2}\right) dx \\
&= \int \sin(u)\left(-\frac{du}{dx}\right) dx \\
&= -\int \sin(u)du \\
&= \cos(x^{-1}) + C
\end{aligned}$$

24. Construct the antiderivative family for the function $y = f(x) = \dfrac{x^2}{\left(x^3+2\right)^2}$.

Solution. Let $u = x^3 + 2$. Then we have

$$\begin{aligned}
\int \frac{x^2}{\left(x^3+2\right)^2} dx &= \int \left(\frac{1}{\left(x^3+2\right)^2}\right)\left(x^2\right) dx \\
&= \int u^{-2}\left(\frac{1}{3}\cdot\frac{du}{dx}\right) dx \\
&= \frac{1}{3}\int u^{-2} du \\
&= -\frac{1}{3\left(x^3+2\right)} + C
\end{aligned}$$

25. Evaluate $\int \left(x^{-1} + 3\frac{\sec^2(x)}{\tan(x)}\right) dx$

Solution. Let $u = \tan(x)$. Then we have

$$\begin{aligned}
\int \left(x^{-1} + 3\frac{\sec^2(x)}{\tan(x)}\right) dx &= \int x^{-1} dx + 3\int \frac{\sec^2(x)}{\tan(x)} dx \\
&= \ln|x| + 3\int \left(\frac{1}{\tan(x)}\right)\left(\sec^2(x)\right) dx \\
&= \ln|x| + 3\int \frac{1}{u}\left(\frac{du}{dx}\right) dx \\
&= \ln|x| + 3\ln|\tan(x)| + C
\end{aligned}$$