MATH 1910 EXAM III

100 points

NAME:____________________________

Write down the antiderivative family for each function below. These are worth five points each.

(1) $f(x) = \cos(x)$ $\qquad$ $\int f(x)dx = \sin(x) + C$

(2) $g(y) = \ln(y)$ $\qquad$ $\int g(y)dy = y\ln(y) - y + C$

(3) $h(z) = \dfrac{1}{z}$ $\qquad$ $\int h(z)dz = \ln|z| + C$

(4) $k(a) = a^3$ $\qquad$ $\int k(a)da = \frac{1}{4}a^4 + C$

(5) $j(b) = b^{1/2}$ $\qquad$ $\int j(b)db = \frac{2}{3}b^{3/2} + C$

15 pts **8.** Consider the function $f(x) = \dfrac{1}{x} - \dfrac{1}{x^2}$.

Part (a) What is the derivative for f? You must show your work for full credit.

Solution. Using the power rule, we know

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x^{-1}\right] - \frac{d}{dx}\left[x^{-2}\right] \\ &= -x^{-2} + 2x^{-3} \\ &= \frac{2}{x^3} - \frac{1}{x^2} \\ &= \frac{2-x}{x^3} \end{aligned}$$

Part (b) What are the critical numbers for f? You must show your work for full credit. (Do not classify the crtitical numbers.)

Solution. We know that f' is undefined when $x^3 = 0$; and this occurs when $x = 0$. We know that $f'(x) = 0$ implies $2 - x = 0$; and this occurs when $x = 2$. Therefore, the function f has two critical numbers — $x = 0$, and $x = 2$.

10 pts **9.** A function f and its first derivative are given below.

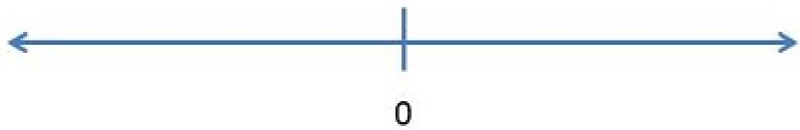
$$f(x) = \ln\left(x^4 + x^2 + 1\right) \qquad f'(x) = \frac{4x^3 + 2x}{x^4 + x^2 + 1}$$

This function has a single critical number at $x = 0$. (You do not have to show this.) Use the First Derivative Test to classify this critical number as a value where f has either a relative maximum output or a relative minimum output. You must show your work for full credit.

Solution. The critical number divides the real number line into two rays, namely $(-\infty, 0]$ and $[0, +\infty)$. To apply the First Derivative Test, we need to select a "test value" from each ray (other than the critical number) and input these test values into the derivative function.

- Letting $x = -1$, we see that $f'(-1) = -2 < 0$; hence, the function f is decreasing on the ray $(-\infty, 0]$.
- Letting $x = 1$, we see that $f'(1) = 2 > 0$; hence, the function f is increasing on the ray $[0, +\infty)$.

We may therefore conclude that f has a relative minimum output when $x = 0$.



10 pts **10.** Find the absolute maximum and minimum values (if they exist) for the function $f(x) = \ln\left(x^4 + x^2 + 1\right)$ on the interval $-1 < x < 2$. You must show your work for full credit.

Solution. The absolute extrema for the function f will occur at critical numbers in the specified interal or at the endpoints of the specified interval. Now,

$$f(-1) = \ln(3) \qquad f(0) = \ln(1) = 0 \qquad f(2) = \ln(21)$$

The function f clearly achieves absolute minimum output when $x = 0$. Since the largest output for f occurs at the endpoint $x = 2$, which is *not* included in the interval, we must conclude that f has no absolute maximum output on the specified interval.

10 pts **11.** Solve the differential equation $y' = 1 + 3x^{1/2}$, if we require $y(4) = 21$. You must show your work for full credit.

Solution. First, observe that the general solution is given by

$$y = \int \left[1 + 3x^{1/2}\right] dx = x + 2x^{3/2} + C$$

Now, we are told that

$$21 = y(4) \Longrightarrow 21 = 4 + 3\,(4)^{3/2} + C \Longrightarrow 21 = 20 + C$$

Therefore, we know that we must let $C = 1$; and the particular solution is $y = x + 3x^{3/2} + 1$.

20 pts **12.** Suppose we want to find numbers m and n, both greater than or equal to 0, whose sum is fixed at 300 and whose product P is *as small as possible.*

(a) What is the optimization formula?

$$P = mn$$

(b) What restrictions do we have?

$$m \geq 0 \qquad n \geq 0 \qquad m + n = 300$$

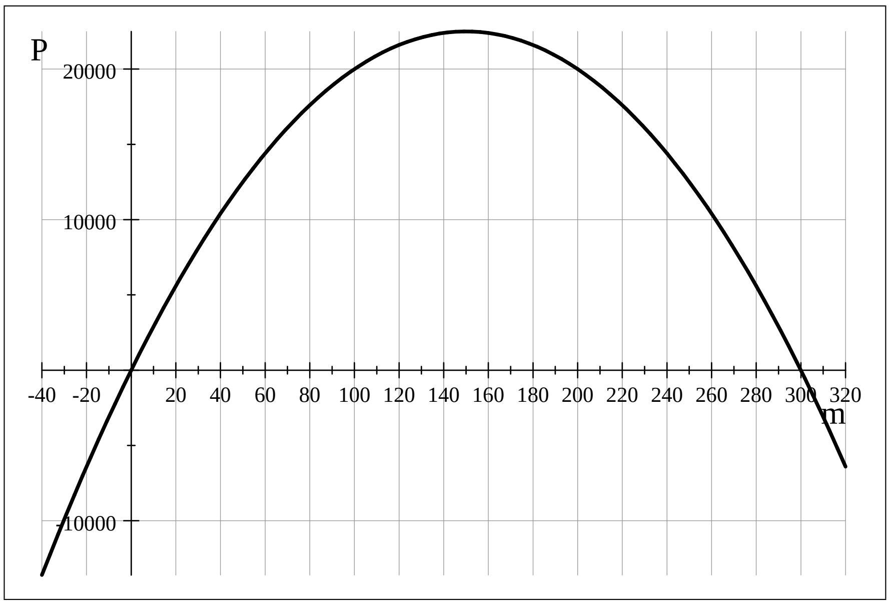
(c) Written in terms of m, what is the optimization function for this problem?

$$m + n = 300 \Longrightarrow n = 300 - m \qquad\qquad P = f(m) = m(300 - m)$$

(d) What is the relevant domain for this function?

$$0 \leq m \leq 300$$

(e) The diagram below shows a graph of the optimization function on an interval that is larger than the relevant domain. Using this graph and the relevant domain, which value or values of m solve the problem?



The relevant domain for the function is the interval $0 \leq m \leq 300$. The smallest value of the product under these restrictions is clearly 0, and this value occurs twice — when $m = 0$ and again when $m = 300$.

10 pts **11.** Find the antiderivative family for the function $f(x) = 3 + \dfrac{2}{x^3} - 5\sin(x)$. You must show your steps for full credit.

$$\begin{aligned}
\int \left[3 + \frac{2}{x^3} - 5\sin(x)\right] dx &= \int 3dx + 2\int x^{-3}dx - 5\int \sin(x)dx \\
&= 3x - x^{-2} + 5\cos(x) + C \\
&= 3x - \frac{1}{x^2} + 5\cos(x) + C
\end{aligned}$$

BONUS:

10 pts A rectangle must have a fixed area of 100 square inches. If we want to construct the rectangle so that that its perimeter is as small as possible, what is the optimization function, written in terms of width only?

Let W and L represent the width and length, respective, of the rectangle, both measured in inches, and let P represent the perimeter of the rectangle, measured in inches. The optimization formula will be

$$P = 2W + 2L$$

Now, we know that $W > 0$, $L > 0$, and $WL = 100$ are the constraints given in the problem. The third constraint tells us that

$$L = \frac{100}{W} \qquad\qquad P = f(W) = 2\left[W + \frac{100}{W}\right]$$