# MATH 1910 PRACTICE EXAM III

1. ____**D**____ Suppose that a function $f$ is increasing and continuous on the half-open interval $(-2, 1]$. Which of the following statements must be true?

   (a) The function $f$ has no absolute extrema.

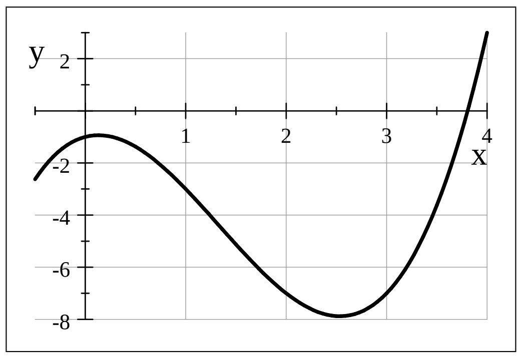
   (b) The function $f$ has an absolute minimum at $x = 1$ and an absolute maximum at $x = -2$.

   (c) The function $f$ has an absolute minimum at $x = -2$ and an absolute maximum at $x = 1$.

   (d) The function $f$ has an absolute maximum at $x = 1$ and no absolute minimum.

   (e) The function $f$ has an absolute minimum at $x = -2$ and no absolute maximum.

2. ____**C**____ Suppose a function $f$ is defined on the segment $(-.5, 4)$ by the graph shown below. Which of the following statements must be true?




   (a) The function $f$ has no absolute extrema.

   (b) The function $f$ has an absolute maximum and an absolute minimum.

   (c) The function $f$ has an absolute minimum but no absolute maximum.

   (d) The function $f$ has an absolute maximum but no absolute minimum.

   (e) The function $f$ has two absolute maxima but no absolute minima.

3. ____**B**____ Suppose that $f$ has a single critical point at $x = 3$. If $f'(0) = -2$ and $f'(4) = 6$, then we know

   (a) $f$ has no relative extrema.

   (b) $f$ has a relative minimum at $x = 3$.

   (c) $f$ has a relative maximum at $x = 3$.

   (d) $f$ has an inflection point at $x = 3$

   (e) $f$ has a horizontal tangent line at $x = 3$.

4. ____**C**____ Suppose that $f\,'(2) = 0$. If $f''(2) = -5$, then we know

   (a) $f$ has no relative extremum at $x = 2$.

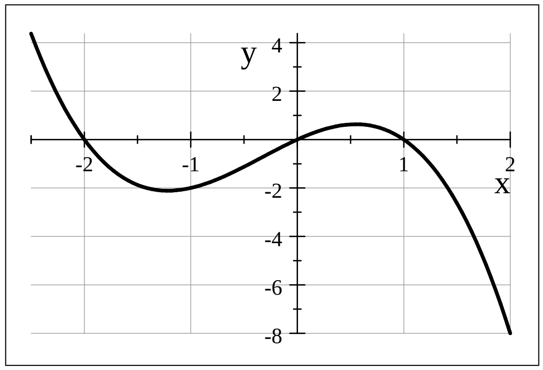
   (b) $f$ has a relative minimum at $x = 2$.

   (c) $f$ has a relative maximum at $x = 2$.

   (d) $f$ has an inflection point at $x = 2$.

   (e) $f$ has a kink in its graph at $x = 2$.

The diagram below shows the *derivative* graph for a function $f$. Use this graph to answer Problems 5, 6, and 7.




5. **A** Based on the *derivative* graph shown above, we see that $f$ has critical points at

(a) $x=-2$, $x=0$, and $x=1$. (b) $x=-1.25$ and $x=.6$.
(c) only $x=.6$. (d) only $x=-1.25$.

(e) $x=-2.5$ and $x=2$.

6. **A** Based on the *derivative* graph shown above, we see that $f$ has relative maxima

(a) at $x=-2$ and $x=1$. (b) only at $x=0$.
(c) only at $x=.6$. (d) only at $x=-1.25$.

(e) at $x=-2.5$ and $x=2$.

7. **B** Based on the *derivative* graph shown above, we know that $f$ will have inflection points at

(a) $x=-2$, $x=0$, and $x=1$. (b) $x=-1.25$ and $x=.6$.
(c) only $x=.6$. (d) only $x=-1.25$.

(e) $x=-2.5$ and $x=2$.

Problems 8 - 10 refer to the function and its derivatives shown below.

$$f(x)=\ln\left(2x^2-2x+1\right) \qquad f'(x)=\frac{2(2x-1)}{2x^2-2x+1} \qquad f''(x)=\frac{8x(1-x)}{\left(2x^2-2x+1\right)^2}$$

8. **A** The function $f$ will have critical points

(a) only at $x=1/2$. (b) at $x=0$ and $x=1$.
(c) at $x=\frac{1}{2}\left(1\pm\sqrt{3}\right)$. (d) at $x=1/2$ and $x=\frac{1}{2}\left(1\pm\sqrt{3}\right)$.

(e) at no value of $x$.

9. **E** The function $f$ will have a relative maximum

(a) only at $x=1/2$. (b) at $x=0$ and $x=1$.
(c) at $x=\frac{1}{2}\left(1\pm\sqrt{3}\right)$. (d) at $x=1/2$ and $x=\frac{1}{2}\left(1\pm\sqrt{3}\right)$.

(e) at no value of $x$.

10. $\underline{\quad\textbf{C}\quad\quad}$ On the interval $[1, 3]$, the function $f$ will have its absolute minimum

(a) $x = 3$ (b) $x = \frac{1}{2}\left(1 + \sqrt{3}\right)$

(c) $x = 1$ (d) $x = 1/2$

(e) $x = 2$

11. Consider the function $f(x) = x + \dfrac{1}{x}$.

(a) Compute the first derivative for $f$.

**Solution.** We know that

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

(b) Identify the critical numbers for $f$. Show your work.

**Solution.** Note that $f'(x)$ is undefined when $x = 0$. Setting $f'(x) = 0$ gives us $x^2 - 1 = 0$. Therefore, we know that $f'(x) = 0$ when $x = \pm 1$.

(c) Use the First Derivative Test to determine which of these critical numbers yield relative maximum or minimum outputs for $f$. Show your steps.

**Solution.** We need to select a test value from each of the sets $(-\infty, -1]$, $[-1, 0]$, $[0, 1]$, and $[1, +\infty)$. Observe

- $f'(-2) = 3/4 > 0$ (The graph of $f$ is increasing on the set $-\infty < x < -1$.)
- $f'(-1/2) = -3 < 0$ (The graph of $f$ is decreasing on the set $-1 < x < 0$.)
- $f'(1/2) = -3 < 0$ (The graph of $f$ is decreasingon the set $0 < x < 1$.)
- $f'(2) = 3/4 > 0$ (The graph of $f$ is increasing on the set $1 < x < +\infty$.)

We may conclude that $f$ has relative maximum output when $x = -1$ and relative minimum output when $x = 1$. The function $f$ does not have a relative extremum at $x = 0$.

12. Solve the differential equation $y'' = x + \cos(x)$ if we also require $y'(0) = -1$ and $y(0) = 3$. You must show your steps for full credit.

**Solution.** Observe

$$y' = \int (x + \cos(x))\, dx \Longrightarrow y' = \frac{x^2}{2} + \sin(x) + C$$

We want to have $y'(0) = -1$; this tells us that $\frac{0^2}{2} + \sin(0) + C = -1$. Consequently, we know that we want to let $C = -1$. Now

$$y = \int \left(\frac{x^2}{2} + \sin(x) - 1\right) dx \Longrightarrow y = \frac{x^3}{6} - \cos(x) - x + D$$

We want to have $y(0) = 3$; this tells us that $\frac{0^3}{6} - \cos(0) - 0 + D = 3$. Consequently, we know that $D = 4$. Therefore, the particular solution we want is

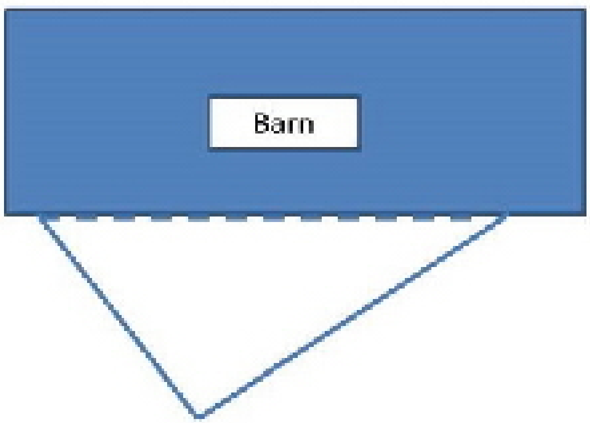
$$y = \frac{x^3}{6} - \cos(x) - x + 4$$

13. Find the antiderivative family for the function $f(x) = 3x^{1/2} - \dfrac{4}{x^4} - 10\sec^2(x)$. You must show your work for full credit.

**Solution.** Observe that

$$\begin{aligned}
\int \left[3x^{1/2} - \frac{4}{x^4} - 10\sec^2(x)\right] dx &= 3\int x^{1/2}dx - 4\int x^{-4}dx - 10\int \sec^2(x)dx \\
&= 3\left(\frac{2}{3}\right)x^{3/2} - 4\left(\frac{1}{-3}\right)x^{-3} - 10\tan(x) + C \\
&= x^{3/2} + \frac{4}{x^3} - 10\tan(x) + C
\end{aligned}$$

14. The area of a right triangle is given by $A = \frac{1}{2}xy$, where $x$ and $y$ represent the lengths of the legs of the triangle. A rancher wants to make a corral in the form of a right triangle adjacent to her barn. The barn will serve as the hypotenuse of the triangle and therefore that side of the corral needs no fencing. If the amount of fencing she has is fixed at 100 feet, and she wants to use all of the fencing, what dimensions will maximize the area of the corral?




**Solution.** Let $x$ and $y$ represent the dimensions of the corral, measured in feet. Let $A$ represent the area of the corral, measured in square feet. We know that the optimization formula will be

$$A = \frac{xy}{2}$$

We also know that $x > 0$ and $y > 0$, since these variable represent dimensions. We also know that $100 = x + y$. This equation tells us that $y = 100 - x$; therefore, we have

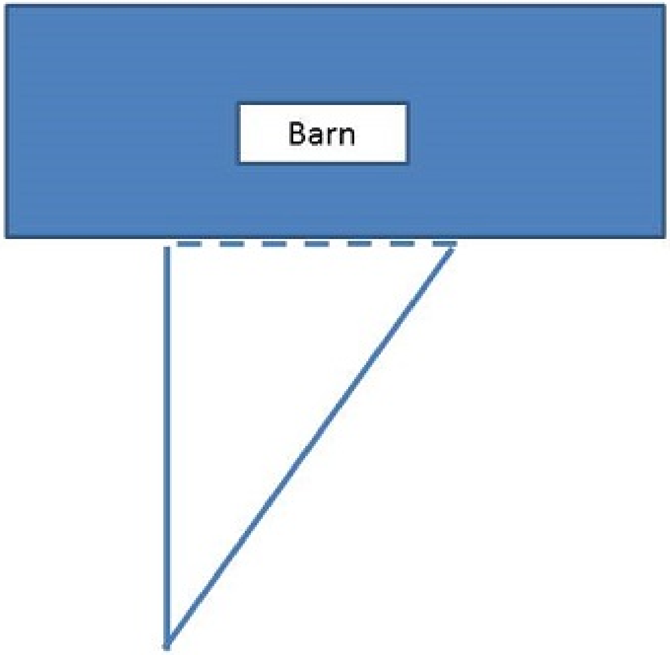
$$A = f(x) = \frac{x(100 - x)}{2} = \frac{100x - x^2}{2}$$

The relevant domain for this function is $0 < x < 100$. Now,

$$f'(x) = 50 - x$$

Therefore, we know that $f$ has a single critical number, namely $x = 50$. Furthermore, since $f''(x) = -1$, we know that $f''(50) < 0$. Consequently, the Second Derivative Test tells us that $f$ has its maximum output when $x = 50$ feet. The dimensions of the corral will be $x = y = 50$ feet.

15. Suppose instead that the rancher wants to orient the corral as shown in the figure below. (One leg is against the barn instead of the hypotenuse.) If she has 100 feet of fencing and wants to use all of it in making the corral, what should the lengths of the legs be to maximize the area of the corral?




**Solution.** Let $x$ and $y$ represent the dimensions of the corral, measured in feet. Let $A$ represent the area of the corral, measured in square feet. We know that the optimization formula will be

$$A = \frac{xy}{2}$$

We also know that $x > 0$ and $y > 0$, since these variable represent dimensions. In this case, one leg and the *hypotenuse* of the triangle must be fenced. Let's assume that $x$ represents the leg of the triangle that needs fencing. The length of the hypotenuse is $\sqrt{x^2+y^2}$. Since the length of the leg and the hypotenuse must add to 100 feet, we know

$$\begin{aligned} 100 = x + \sqrt{x^2+y^2} &\implies (100-x)^2 = x^2+y^2 \\ &\implies 10000 - 200x = y^2 \\ &\implies 10\sqrt{100-2x} = y \end{aligned}$$

Therefore, the optimization function will be

$$A = f(x) = 5x\sqrt{100-2x}$$

The relevant domain will be $0 < x < 50$, since we must have $100 - 2x > 0$ in order for $y$ to be defined and positive. Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[5x\sqrt{100-2x}\right] \\ &= \frac{d}{dx}[5x]\sqrt{100-2x} + 5x\frac{d}{dx}\left[\sqrt{100-2x}\right] \\ &= 5\sqrt{100-2x} + (5x)\left(\frac{-2}{2\sqrt{100-2x}}\right) \\ &= \frac{5\left(\sqrt{100-2x}\right)^2 - 5x}{\sqrt{100-2x}} \\ &= \frac{500-15x}{\sqrt{100-2x}} \end{aligned}$$

Notice that $f'(x)$ is undefined when $x = 50$, but this value of $x$ does not lie in the relevant domain. Now, setting $f'(x) = 0$ tells us

$$500 - 15x = 0 \Longrightarrow x = \frac{100}{3} \approx 33.33 \text{ feet}$$

The second derivative of $f$ is difficult to compute, so we will resort to the first derivative test to analyze the behavior of $f$ at this critical number. We pick two test values, one to the left of $x = 100/3$ and one between $x = 100/3$ and $x = 50$ (the other critical number for $f$). Let's try $x = 0$ and $x = 40$ as test values. Observe

- $f'(0) = \frac{500}{10} > 0$ (Hence we know that the graph of $f$ is increasing to the left of $x = 100/3$.)
- $f'(40) = -\frac{100}{\sqrt{20}} < 0$ (Hence we know that the graph of $f$ is decreasing between $x = 100/3$ and $x = 50$.)

Consequently, we may conclude that the area of the corral is maximized when $x = 100/3$ feet. The length of the unfenced leg would be

$$y = 10\sqrt{100 - 2\left(\frac{100}{3}\right)} = \frac{100}{3} \text{ feet}$$