MATH 1920 EXAM II

100 points

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10 pts 1. Use integration by parts to find the antiderivative family for $f(x) = x\cos(x)$. You must show your steps for full credit.

$$\begin{aligned} \int x\cos(x)dx &= uv - \int vdu \qquad \text{Let} \begin{cases} u = x \text{ so } du = 1 \\ dv = \cos(x)dx \text{ so } v = \sin(x) \end{cases} \\ &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) + \cos(x) + C \end{aligned}$$

15 pts 2. Use Integration by Parts to evaluate the definite integral $\int_1^2 x\ln(x)dx$. You must show your work for full credit.

Solution. There are two ways to evaluate this definite integral, and both rely on integration by parts.

$$\begin{aligned} \int_1^2 x\ln(x)dx &= uv\big|_{x=1}^{x=2} - \int_{x=1}^{x=2} vdu \qquad \text{Let} \begin{cases} u = \ln(x) \text{ so } du = \frac{1}{x} \\ dv = xdx \text{ so } v = \frac{x^2}{2} \end{cases} \\ &= \frac{x^2\ln(x)}{2}\bigg|_{x=1}^{x=2} - \frac{1}{2}\int_{x=1}^{x=2} x^2\left(\frac{1}{x}\right)dx \\ &= \frac{x^2\ln(x)}{2}\bigg|_{x=1}^{x=2} - \frac{x^2}{4}\bigg|_{x=1}^{x=2} \\ &= [2\ln(2) - 0] - \frac{1}{4}[4-1] \\ &= \frac{8\ln(2) - 3}{4} \end{aligned}$$

$$\begin{aligned} \int_1^2 x\ln(x)dx &= uv\big|_{x=1}^{x=2} - \int_{x=1}^{x=2} vdu \qquad \text{Let} \begin{cases} u = x \text{ so } du = 1 \\ dv = \ln(x)dx \text{ so } v = x\ln(x) - x \end{cases} \\ &= x(x\ln(x) - x)\big|_{x=1}^{x=2} - \int_{x=1}^{x=2} (x\ln(x) - x)dx \\ &= x(x\ln(x) - x)\big|_{x=1}^{x=2} + \int_{x=1}^{x=2} xdx - \int_{x=1}^{x=2} x\ln(x)dx \end{aligned}$$

At this point, notice that we have encountered a circular integral. The last equality tells us

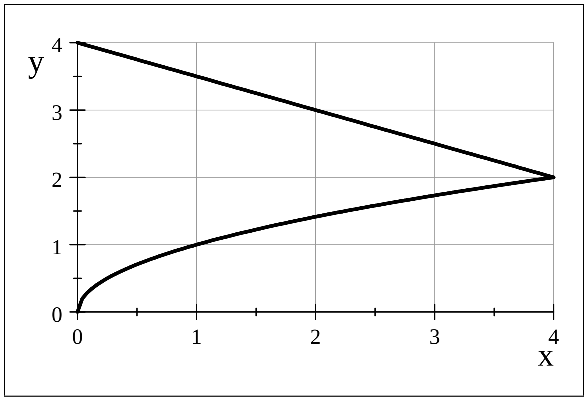
$$\begin{aligned} 2\int_1^2 x\ln(x)dx &= x(x\ln(x) - x)\big|_{x=1}^{x=2} + \int_{x=1}^{x=2} xdx \\ &= x(x\ln(x) - x)\big|_{x=1}^{x=2} + \frac{x^2}{2}\bigg|_{x=1}^{x=2} \\ &= [(4\ln(2) - 4) - (2(0) - 1)] + \frac{1}{2}[4-1] \\ &= 4\ln(2) - 3 + \frac{3}{2} \\ &= \frac{8\ln(2) - 3}{2} \end{aligned}$$

Dividing both sides of the equation by 2 gives us the final answer.

10 pts 3. Use an appropriate substitution to determine the value of the definite integral $\int_1^2 x\sin(\pi x^2)dx$. You must show your work for full credit.

$$\begin{aligned}\int_1^2 x\sin(\pi x^2)dx &= \frac{1}{2\pi}\int_{u=\pi}^{u=4\pi}\sin(u)du \qquad \text{Let } u=\pi x^2 \text{ so } \frac{du}{dx}=2\pi x\\ &= -\frac{1}{2\pi}\cos(u)\Big|_{u=\pi}^{u=4\pi}\\ &= -\frac{1}{\pi}\end{aligned}$$

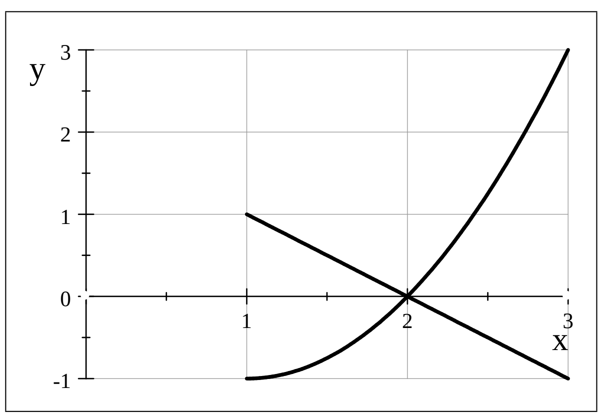
10 pts 4. Let $\mathcal{R}$ be the region between the curves $f(x)=\sqrt{x}$ and $g(x)=4-\frac{x}{2}$ on the interval $0\le x\le 4$. Let V represent the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis.



SET UP the definite integral that computes V using the Method of Washers. DO NOT EVALUATE.

$$V=\pi\int_0^4\left[\left(4-\frac{x}{2}\right)^2-x\right]dx$$

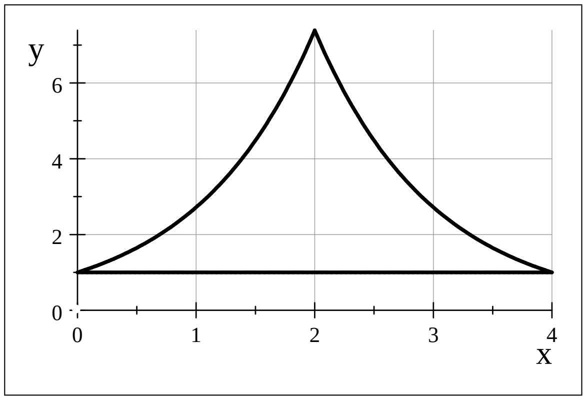
10 pts 5. Let $\mathcal{R}$ be the region bounded by the curves $f(x)=(x-1)^2$ and $g(x)=2-x$ on the interval $1\le x\le 3$. Let V be the volume of the solid obtained by revolving $\mathcal{R}$ about the line $y=-1$.



SET UP the definite integral that computes V using the Method of Washers. DO NOT EVALUATE.

$$V=\pi\int_1^2\left[(3-x)^2-\left(1+(x-1)^2\right)^2\right]dx+\pi\int_2^3\left[\left(1+(x-1)^2\right)^2-(3-x)^2\right]dx$$

6. Let $\mathcal{R}$ be the region bounded by the curves $y=e^x$ and $y=e^{4-x}$ and $y=1$ on the interval $0\le x\le 4$ as shown below.



10 pts (a) SET UP the definite integral that computes the area of $\mathcal{R}$ *with respect to the x-axis.* DO NOT EVALUATE.

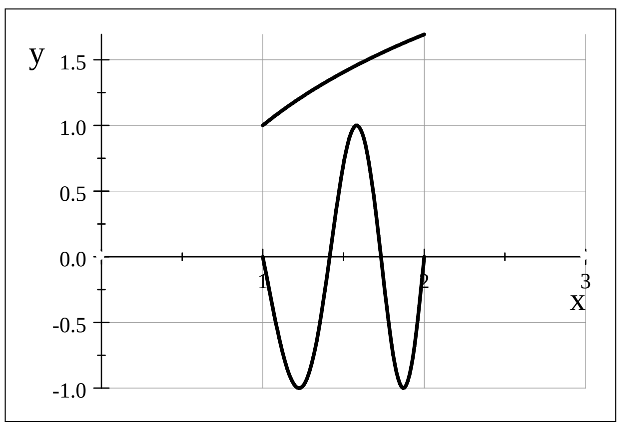
$$A = \int_0^2 (e^x - 1)\, dx + \int_2^4 \left(e^{4-x} - 1\right) dx$$

15 pts (b) SET UP the definite integral that computes the area of $\mathcal{R}$ *with respect to the y-axis.* DO NOT EVALUATE.

Solution. First, note that $y = e^x$ implies $x = \ln(y)$ and $y = e^{4-x}$ implies $x = 4 - \ln(y)$. Now, the region is bounded between these curves on the interval $1 \le y \le e^2$; therefore, we know

$$A = \int_1^{e^2} (4 - 2\ln(y))\, dy$$

10 pts 7. Let $\mathcal{R}$ be the region bounded by the curves $f(x) = 1 + \ln(x)$ and $g(x) = \sin(\pi x^2)$ on the interval $1 \le x \le 2$.



Let V be the volume of the solid obtained by revolving $\mathcal{R}$ about the y-axis. SET UP the integral required to compute V if the Method of Shells is used. DO NOT EVALUATE.

$$V = 2\pi \int_1^2 x\left[1 + \ln(x) - \sin(\pi x^2)\right] dx$$

10 pts 8. Evaluate the definite integral you set up in Problem 7. You must show your steps for full credit. You

may use the results from previous problems on this exam.

$$\begin{aligned}
V &= 2\pi\int_1^2 x dx + 2\pi\int_1^2 x\ln(x)dx - 2\pi\int_1^2 x\sin(\pi x^2)dx \\
&= 2\pi\left[\frac{x^2}{2}\Big|_{x=1}^{x=2} + \frac{8\ln(2)-3}{4} + \frac{1}{\pi}\right] \\
&= 2\pi\left[\frac{3}{2} + \frac{8\ln(2)-3}{4} + \frac{1}{\pi}\right] \\
&= 3\pi + \left(\frac{8\ln(2)-3}{2}\right)\pi + 1 \\
&= \frac{3\pi + 2 + 8\pi\ln(2)}{2}
\end{aligned}$$