MATH 1920 EXAM III

100 points

NAME:____________________________

5 pts each 1. Match each of the indefinite integrals below with the technique best suited for evaluating it. Some options may be used more than once, and others not at all.

(a) B $\int \frac{x+3}{x-x^2}dx$

(b) C $\int \sqrt{2-9x^2}dx$ Let $\sin(\theta) = \frac{3x}{\sqrt{2}}$

(c) D $\int \frac{1}{3x-1}dx$ Let $u = 3x-1$

(d) D $\int \frac{12x^2}{\sqrt{1+4x^3}}dx$ Let $u = 1+4x^3$

(e) B, or C, or D $\int \frac{x}{(x-1)(x+1)}dx$

(f) A $\int x^2 \sin(x)dx$

Note that $\frac{x}{(x-1)(x+1)} = -\frac{x}{1-x^2}$.
Let $u = 1-x^2$ or $\sin(\theta) = x$

(A) Integration by Parts (B) Partial Fraction Decomposition

(C) Trigonometric Substitution (D) Direct u Substitution

10 pts 2. By making an appropriate trigonometric substitution, show that $\int \frac{x^3}{\sqrt{1-x^2}}dx = \int \sin^3(\theta)d\theta$. Show your steps, but do not integrate.

Let $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ be such that $\sin(\theta) = x$. This tells us $\theta = \arcsin(x)$ and $\cos(\theta)d\theta = dx$. Therefore,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}dx &= \int \frac{\sin^3(\theta)}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)d\theta \\
&= \int \frac{\sin^3(\theta)}{\cos(\theta)}\cos(\theta)d\theta \\
&= \int \sin^3(\theta)d\theta
\end{aligned}$$

10 pts 3. Find the antiderivative family for $f(\theta) = \sin^3(\theta)$. You must show your steps for full credit.

$$\begin{aligned}
\int \sin^3(\theta)d\theta &= \int \sin^2(\theta)\sin(\theta)d\theta \\
&= \int (1-\cos^2(\theta))\sin(\theta)d\theta \\
&= \int \sin\theta d\theta - \int \cos^2(\theta)\sin\theta d\theta \qquad \text{Let } u = \cos(\theta) \text{ so } du = -\sin(\theta)d\theta \\
&= \int \sin\theta d\theta + \int u^2 du \\
&= -\cos(\theta) + \frac{\cos^3(\theta)}{3} + C
\end{aligned}$$

10 pts 4. After making the substitution $\tan(\theta) = 2x$, it can be shown that

$$\int x^3\sqrt{1+4x^2}dx = \frac{1}{120}\left[3\sec^5(\theta) - 5\sec^3(\theta)\right] + C$$

Complete the integration process by rewriting the antiderivative family as functions of x instead of θ. You must show your work for full credit.

(a) Since θ is an acute angle in a right triangle, we know that $\tan(\theta) = 2x$ tells us the Side Opposite θ in this triangle is $2x$, while the Side Adjacent θ in this triangle is 1. Therefore, the hypotenuse of this triangle is $\sqrt{1+4x^2}$; and this tells us $\sec(\theta) = \sqrt{1+4x^2}$. Therefore,

$$\begin{aligned}\int x^3\sqrt{1+4x^2}dx &= \frac{1}{120}\left[3\sec^5(\theta) - 5\sec^3(\theta)\right] + C \\ &= \frac{1}{120}\left[3(1+4x^2)^{5/2} - 5(1+4x^2)^{3/2}\right] + C\end{aligned}$$

15 pts 5. Construct the partial fraction decomposition for the rational function $f(x) = \dfrac{x+3}{x-x^2}$. You must show your steps for full credit. (There is no integration involved here.)

$$\begin{aligned}\frac{x+3}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} \quad &\Longrightarrow \quad x+3 = A(1-x) + Bx \\ &\Longrightarrow \quad x+3 = (B-A)x + A\end{aligned}$$

Equating coefficients tells us that $A = 3$ and $B - A = 1$. It follows that $B = 4$. Therefore,

$$\frac{x+3}{x-x^2} = \frac{3}{x} + \frac{4}{1-x}$$

15 pts 6. It can be shown that

$$f(x) = \frac{3x^4 - x^3 + 8x^2 - 5x + 3}{3x^3 - x^2 + 3x - 1} = x + \frac{2}{3x-1} + \frac{x-1}{1+x^2}$$

Use this fact to find the antiderivative family for the function f.

$$\begin{aligned}\int f(x)dx &= \int xdx + 2\int \frac{1}{3x-1}dx + \int \frac{x}{1+x^2}dx - \int \frac{1}{1+x^2}dx \qquad \text{Let } u = 3x-1 \text{ and let } v = 1+x^2 \\ &= \frac{x^2}{2} + \frac{2}{3}\ln|3x-1| + \frac{1}{2}\ln(1+x^2) - \arctan(x) + C\end{aligned}$$

10 pts 7. Evaluate the convergent integral $\displaystyle\int_1^{+\infty} \frac{1}{x^3}dx$. You must show your steps for full credit.

$$\int x^{-3}dx = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$$

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x^3}dx &= \lim_{h\longrightarrow+\infty}\left[-\frac{1}{2x^2}\right]_1^h \\ &= -\frac{1}{2}\lim_{h\longrightarrow+\infty}\left[\frac{1}{h^2} - 1\right] \\ &= \frac{1}{2}\end{aligned}$$