MATH 1920 EXAM IV

100 points

NAME:____________________________

10 pts 1. Do either of the following geometric series converge? Explain your answer.

$$\text{(a)}\quad \sum_{j=10}^{\infty}\left(\frac{3}{5}\right)^j \qquad\qquad \text{(b)}\quad \sum_{j=0}^{\infty}\left(-\frac{5}{3}\right)^j$$

Series (a) will converge because $3/5 < 1$; however, Series (b) will diverge because $-5/3 < -1$.

10 pts 2. Compute the value of the geometric series $\sum_{j=0}^{\infty}(-1)^j \frac{2^j}{3^{j+1}}$. You must show your steps for full credit.

$$\sum_{j=0}^{\infty}(-1)^j \frac{2^j}{3^{j+1}} = \sum_{j=0}^{\infty}\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)^j = \frac{1/3}{1+2/3} = \frac{1}{5}$$

10 pts 3. Suppose a function f has the following third-degree Taylor polynomial approximation.

$$T_3(x,-2) = 5 - 7(x+2) + 12(x+2)^2 - \frac{13}{3}(x+2)^3$$

(a) What is the approximate value of $f(1)$? Show your work.

$$\begin{aligned} f(1) &\approx 5 - 7(1+2) + 12(1+2)^2 - \frac{13}{3}(1+2)^3 \\ &= 5 - 21 + 108 - 117 \\ &= -25 \end{aligned}$$

(b) What is the value of $f^{(3)}(-2)$? Show your work.

$$-\frac{13}{3} = \frac{f^{(3)}(-2)}{6!} \Longrightarrow f^{(3)}(-2) = -26$$

4. Consider the function $f(x) = \dfrac{1}{4+3x}$. We can rewrite this function as

$$f(x) = \frac{1/4}{1+(3x/4)}$$

10 pts (a) Construct a series representation for f.

$$f(x) = \sum_{j=0}^{\infty}\left(\frac{1}{4}\right)\left(-\frac{3x}{4}\right)^j = \sum_{j=0}^{\infty}(-1)^j \frac{3^j}{4^{j+1}}x^j$$

10 pts (b) What is the interval of convergence for this series representation?

$$-1 < \frac{3x}{4} < 1 \qquad \text{or, equivalently,} \qquad -\frac{4}{3} < x < \frac{4}{3}$$

15 pts 5. It can be shown that, so long as $3 < x < 5$, a series representation for $f(x) = \dfrac{1}{5-x}$ is

$$f(x) = \sum_{j=0}^{\infty} (x-4)^j$$

Use the fact that $\ln(5-x) = -\int_4^x \frac{1}{5-x}dx$ to construct a possible series representation for $g(x) = \ln(5-x)$.

$$\begin{aligned} \ln(5-x) &= -\int_4^x \sum_{j=0}^{\infty} (t-4)^j \, dt \\ &\sim -\sum_{j=0}^{\infty} \int_4^x (t-4)^j \, dt \\ &= -\sum_{j=0}^{\infty} \frac{(x-4)^{j+1}}{j+1} \end{aligned}$$

15 pts 6. Consider the function $f(x) = \sin^2(x)$. With patience and the double angle formulas, it can be shown that

- $f'(x) = \sin(2x)$
- $f''(x) = 2\cos(2x)$
- $f^{(3)}(x) = -4\sin(2x)$
- $f^{(4)}(x) = -8\cos(2x)$
- $f^{(5)}(x) = 16\sin(2x)$

Use this information to construct $T_4(x, \pi)$ — the fourth degree Taylor polynomial for f centered at $x = \pi$.

$$\begin{aligned} T_4(x,\pi) &= f(\pi) + \frac{f'(\pi)}{1}(x-\pi) + \frac{f''(\pi)}{2}(x-\pi)^2 + \frac{f^{(3)}(\pi)}{6}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{24}(x-\pi)^4 \\ &= (x-\pi)^2 - \frac{1}{3}(x-\pi)^4 \end{aligned}$$

7. The Taylor Series for $f(x) = \ln(x)$ centered at $x = 2$ is given by

$$\ln(2) + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(x-2)^j}{j \cdot 2^j}$$

10 pts (a) What is the radius of convergence for this series?

$$R = \lim_{j\to+\infty} \frac{j \cdot 2^j}{(j+1)\cdot 2^{j+1}} = 2 \lim_{j\to+\infty} \frac{j}{j+1} = 2(1) = 2$$

10 pts (b) What is the interval of convergence for this series?

$$|x-2| < 2 \Longrightarrow 0 < x < 4$$