MATH 1920 PRACTICE EXAM IV

100 points

NAME:________________

(1) Evaluate the series $\sum_{j=4}^{+\infty} \frac{1}{4^j}$. Show your work.

(2) Evaluate the series $\sum_{j=0}^{+\infty} \frac{1-2^j}{3^j}$. Show your work.

(3) Suppose that a function f is such that $f(1) = 2$, $f'(1) = -3$, and $f''(1) = 4$.

(a) Use this information to construct the second-degree Taylor polynomial for f centered at $x = 1$.

(b) Use Part (a) to estimate the value of $f(2)$.

(4) Consider the function $f(x) = \dfrac{1}{4-5x}$.

(a) Show that $f(x) = \dfrac{1}{4}\sum_{j=0}^{+\infty}\left(\dfrac{5}{4}\right)^j x^j$ when $|x| < 4/5$.

(b) Use the fact that $4 - 5x = 1 - (5x - 3)$ to help show $f(x) = \sum_{j=0}^{\infty} 5^j (x - 3/5)^j$ for $2/5 < x < 4/5$.

(c) Use Parts (a) and (b) and the fact that

$$\ln(4-5x) - \ln(4) = -5\int_0^x \frac{1}{4-5t}\,dt \qquad \text{and} \qquad \ln(4-5x) = -5\int_{3/5}^x \frac{1}{4-5t}\,dt$$

to help construct two *possible* series representations for $g(x) = \ln(4-5x)$, one centered at $x = 0$, and the other centered at $x = 3/5$.

(5) Here are the first five derivatives for $g(x) = \ln(4-5x)$.

$$f'(x) = \frac{5}{5x-4} \qquad f''(x) = -\frac{25}{(4-5x)^2} \qquad f^{(3)}(x) = \frac{250}{(5x-4)^3}$$

$$f^{(4)}(x) = -\frac{3750}{(4-5x)^4} \qquad f^{(5)}(x) = \frac{75000}{(5x-4)^5}$$

(a) Use this information to construct the fifth-degree Taylor polynomial for g centered at $x = 0$.

(b) Use the series centered at $x = 0$ you constructed in the previous problem to create a fifth-degree polynomial approximation for the function g. How does this polynomial compare to the Taylor polynomial you constructed in Part (a)?

(6) Here are the first five derivatives for the function $f(x) = \arcsin(x)$.

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$f^{(3)}(x) = \frac{2x^2+1}{(1-x^2)^{5/2}} \qquad f^{(4)}(x) = \frac{6x^3+9x}{(1-x^2)^{7/2}}$$

$$f^{(5)}(x) = \frac{24x^4+72x^2+9}{(1-x^2)^{9/2}}$$

(a) Use this information to construct the fifth-degree Taylor polynomial for the function f centered at $x = 0$.

(b) Use this information to construct the fifth-degree Taylor polynomial for the function f centered at $x = 1/2$. (Remember that $\arcsin(1/2) = \pi/6$.)

(7) Consider the following functions

$$f(x) = \frac{x}{1+x^4} \qquad g(x) = \arctan(x^2)$$

Observe that $g(x) = 2\int_0^x f(t)\,dt$.

(a) Working from the basic geometric series, construct a series representation for $h(x) = \dfrac{1}{1+x^4}$.

(b) Use your answer from Part (a) to construct a series representation for the function f. (Don't make this too hard.) What is the interval of convergence for this series representation?

(c) Use your answer from Part (b) to construct a *possible* series representation for the function g.

(8) Consider the function $f(x) = \cos^3(x)$. It can be shown that

- $f'(x) = 9\sin^3(x) - 3\sin(x)$
- $f''(x) = 24\cos(x) - 27\cos^3(x)$
- $f^{(3)}(x) = 51\sin(x) - 81\sin^3(x)$
- $f^{(4)}(x) = 243\cos^3(x) - 192\cos(x)$

Use this information to construct the fourth-degree Taylor polynomial for $f(x) = \cos^3(x)$ centered at $x = \pi$.

(9) Find the radius and interval of convergence for the series $\displaystyle\sum_{j=2}^{+\infty} \frac{1}{3^j}(x-2)^j$.

(10) The Taylor Series for the function $f(x) = \ln(x)$ centered at $x = 5$ is given by

$$\ln(5) + \sum_{j=1}^{+\infty} (-1)^{j+1} \frac{(x-5)^j}{j \cdot 5^j}$$

What is the radius and interval of convergence for this power series?

Answers.

(1) We have $\displaystyle\sum_{j=4}^{+\infty}\frac{1}{4^j}=\frac{1}{192}$.

(2) We have $\displaystyle\sum_{j=0}^{+\infty}\frac{1-2^j}{3^j}=-\frac{3}{2}$.

(3) We have $T_2(x,1)=2-3(x-1)+2(x-1)^2$. Also, we know $f(2)\approx 2-3(2-1)+2(2-1)^2=1$.

(4) For Part (c), we can use term-by-term integration to obtain the following possible series representations.

$$\ln(4-5x)\sim\ln(4)-\sum_{j=0}^{\infty}\left(\frac{5}{4}\right)^{j+1}\frac{x^{j+1}}{j+1}\qquad\qquad \ln(4-5x)\sim-\sum_{j=0}^{\infty}\left(\frac{5^{j+1}}{j+1}\right)(x-3/5)^{j+1}$$

(5) The fifth-degree Taylor polynomial is

$$T_5(x,0)=\ln(4)-\left(\frac{5x}{4}+\frac{25x^2}{32}+\frac{125x^3}{192}+\frac{625x^4}{1024}+\frac{625x^5}{1024}\right)$$

Oddly enough, the coefficients of the fourth and fifth-degree terms are the same.

(6) We have

$$T_5(x,0)=x+\frac{x^3}{6}+\frac{3x^5}{40}$$

$$T_5(x,1/2)=\frac{\pi}{6}+\frac{2\sqrt{3}}{3}(x-1/2)+\frac{2\sqrt{3}}{9}(x-1/2)^2+\frac{8\sqrt{3}}{27}(x-1/2)^3+\frac{28\sqrt{3}}{81}(x-1/2)^4+\frac{608\sqrt{3}}{1215}(x-1/2)^5$$

(7) We know that $h(x)=\displaystyle\sum_{j=0}^{\infty}(-1)^jx^{2j}$ and $f(x)=\displaystyle\sum_{j=0}^{\infty}(-1)^jx^{2j+1}$.

$$\arctan(x^2)\sim\sum_{j=0}^{\infty}(-1)^j\frac{x^{2j+2}}{j+1}$$

(8) We have $T_4(x,\pi)=-1+\dfrac{3}{2}(x-\pi)^2-\dfrac{7}{8}(x-\pi)^4$.

(9) We know that $A_j=1/3^j$ and $A_{j+1}=1/3^{j+1}$. Therefore, we know

$$R=\lim_{j\to+\infty}\frac{1/3^j}{1/3^{j+1}}=\frac{1}{3}$$

The interval of convergence will be $5/3<x<7/3$.

(10) We know $A_j=\dfrac{(-1)^{j+1}}{j5^j}$ and $A_{j+1}=\dfrac{(-1)^{j+2}}{(j+1)5^{j+1}}$. Therefore,

$$R=\lim_{j\to+\infty}\frac{(j+1)5^{j+1}}{j5^j}=5\lim_{j\to+\infty}\frac{j+1}{j}=5$$

The interval of convergence will be $0<x<10$.