Cubics, Quartics, and the Dawn of Complex Numbers

Students of algebra are all familiar with the so-called *quadratic formula* which provides the solutions to any quadratic equation. Very few people, however, know that there are similar methods for solving cubic and quartic equations as well. In these notes, we will introduce these esoteric methods and show how they inspired the creation of complex numbers.

Part I: The Cubic Equation

The solution to the cubic equation, popularly attributed to Girolamo Cardano (1501-1576) was actually stolen from Nicolo Tartaglia (1500-1557) by Cardano. Many believe, however, the Tartaglia himself was at least influenced by the work of Scipio Del Ferro who appears to have been the first to obtain a solution. Still, the solution we present is known today as *Cardano's Formula*.

Consider the general cubic equation $x^3 + Ax^2 + Bx + C = 0$, where A, B, C are arbitrary real numbers. The square term in the left-hand side can always be removed by making the variable substitution $x = y - A/3$. Performing this substitution results in an equation of the form

$$y^3 + py + q = 0,$$

where $p = B - \dfrac{A^2}{3}$ and $q = 2\left(\dfrac{A}{3}\right)^3 - B\left(\dfrac{A}{3}\right) + C$.

The polynomial in y above is called a *defective* cubic (because it is "missing" its square term). Since we can always reduce a cubic polynomial to a defective cubic polynomial, we need only concern ourselves with solving defective cubic equations. Sixteenth century mathematicians solved this type of cubic equation by making the (rather obscure) observation that the following equation holds for all real numbers a and b:

$$(a - b)^3 + 3ab(a - b) + (b^3 - a^3) = 0$$

In particular, they observed that this equation "looks" like the defective cubic equation in y. If it is possible to find values for a and b such that

- $q = b^3 - a^3$, and
- $p = 3ab$,

then the equation in a and b can be rewritten as

$$(a - b)^3 + p(a - b) + q = 0$$

Consequently, if values for a and b can be so found, then letting $y = a - b$ will solve the defective cubic equation. Finding the appropriate values for a and b is therefore our goal. Tartaglia used the following method (adjusted to fit modern notation and conventions).

(1) First, square $q = b^3 - a^3$ to obtain $q^2 = b^6 - 2a^3b^3 + a^6$.
(2) Second, cube $p = 3ab$ to obtain $p^3 = 27a^3b^3$.
(3) Third, rewrite $p^3 = 27a^3b^3$ as $\dfrac{4p^3}{27} = 4a^3b^3$.
(4) Add this last equation to $q^2 = b^6 - 2a^3b^3 + a^6$ to obtain

$$q^2 + \frac{4p^3}{27} = b^6 + 2a^3b^3 + a^6 = (b^3 + a^3)^2.$$

The last equation tells us that

$$b^3 + a^3 = \pm\sqrt{q^2 + \frac{4p^3}{27}}.$$

In the sixteenth century, few people accepted the existence of negative numbers as entities in their own right; in keeping with this, Tartaglia would not have considered the negative root to be a solution. He used only the equation

$$b^3 + a^3 = \sqrt{q^2 + \frac{4p^3}{27}}.$$

At this point, Tartaglia had what amounts to a system of two equations in two unknowns:

(1) $a^3 - b^3 = -q$

(2) $a^3 + b^3 = \sqrt{q^2 + \frac{4p^3}{27}}$.

Using modern algebra, it is easy enough to see that this system gives us the following solutions for a and b:

(1) $a = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

(2) $b = \sqrt[3]{\frac{1}{2}\left(q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$.

Consequently, letting $y = a - b$ gives us the solution to the defective cubic equation known today as *Cardano's Formula*:

For any constants p and q, the solution to the equation $y^3 + py + q = 0$ is given by

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

EXAMPLE 1. Use Cardano's Formula to a solution of the cubic equation $x^3 + 3x^2 + 5x + 2 = 0$.

Letting $x = y - 1$ gives us the corresponding defective cubic $y^3 + 2y - 1 = 0$. Using Cardano's Formula, we find that

$$y = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{8}{27}}} - \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{8}{27}}}.$$

Hence, we know that

$$y = \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{59}{3}}} - \sqrt[3]{-\frac{1}{2} + \frac{1}{6}\sqrt{\frac{59}{3}}} \approx .4533976515.$$

Thus, the solution to the original equation is $x \approx 1.4533976515$.

EXAMPLE 2. Use Cardano's Formula to find a solution of the cubic equation $x^3 + 3x^2 - x - 2 = 0$.

Letting $x = y - 1$ gives us the corresponding defective cubic $y^3 - 4y + 1$. Thus, in this case, $p = -4$ and $q = 1$. Using Cardano's Formula, we find a root for the equation is given by

$$y = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{64}{27}}} - \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{64}{27}}}.$$

Hence, in this case, we obtain

$$y = \sqrt[3]{-\frac{1}{2} + \frac{1}{6}\sqrt{\frac{229}{3}}\,i} - \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{229}{3}}\,i}.$$

The solution of the original cubic that Cardano's Formula gives us is therefore

$$x = \sqrt[3]{-\frac{1}{2} + \frac{1}{6}\sqrt{\frac{229}{3}}\,i} - \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{229}{3}}\,i} + 1.$$

PART II: Complex Numbers

Example 2 in the last section was particularly vexing to sixteenth century mathematicians because, as a quick check on the graphing calculator will show, the polynomial $p(x) = x^3 + 3x^2 - x - 3$ has three *real* roots. Hence, the strange difference of complex cube roots we obtained for y above must be a real number.

The difficulty, of course, lies in understanding how to take cube roots of complex numbers. The ability to do this was quite beyond sixteenth century mathematicians; indeed, no consistent theory of complex numbers existed at that time and most mathematicians of that century refused to consider them at all.

Rafael Bombelli (1526-1572) was one of the first people to attempt an understanding of complex roots. His research was highly specific and did not extend far beyond the scope of one problem; still, what he accomplished helped pave the way for a rigorous theory of complex numbers in the nineteenth century.

Bombelli considered the equation $x^3 - 15x - 4 = 0$. This equation has three real roots, namely $x = 4$ and $x = -2 \pm \sqrt{3}$. However, as you can check, Cardano's formula yields a troublesome difference of complex roots:

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}} = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Bobmelli, who was the first person to let i denote $\sqrt{-1}$, had the idea that, if this sum of complex roots is itself real, then it seems reasonable to let

$$\sqrt[3]{2 + \sqrt{-121}} = a + bi, \quad \text{and} \quad \sqrt[3]{2 - \sqrt{-121}} = a - bi.$$

This idea was a stroke of inspired genius. Bombelli was the first person to introduce the now-standard $a + bi$ notation for complex numbers and the first to use the concept of conjugate pairs. Proceeding now under his notational assumption that $i = \sqrt{-1}$, he took the first equation and cubed it, obtaining

$$2 + 11i = a(a^2 - 3b^2) + b(3a^2 - b^2)i.$$

He then proceeded to equate coefficients, obtaining two equations:

(1) $2 = a(a^2 - 3b^2)$
(2) $11 = b(3a^2 - b^2)$.

This system of equations has only one solution in the integers, namely $a = 2$ and $b = 1$. Hence, Bombelli set

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + i, \quad \text{and} \quad \sqrt[3]{2 - \sqrt{-121}} = 2 - i.$$

Of course, adding these together gives one of the real roots, namely $x = 4$.

Unfortunately, Bombelli's somewhat ad-hoc approach does not work in general, since we have no guarantee that complex cube roots will give rise to complex numbers having integer coefficients. In the case of Example 2 above, Bombelli's approach tells us almost nothing since he provides no systematic way of finding the coefficients a and b if we cannot assume both are integers.

Francois Vieta (1540-1603) used the then-infant field of trigonometry to avoid complex roots altogether. In Cardano's Formula, the expression

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}$$

is called the *discriminant* of the corresponding defective cubic. Cardano's formula contains complex cube roots if and only if this discriminant is negative. In the event that $\Delta < 0$, we must have $p < 0$; hence, in this case, we can rewrite the discriminant as

$$\Delta = \frac{q^2}{4} - \frac{P^3}{27},$$

where $P = |\,p\,|$. Vieta considered this case.

Suppose that $x^3 + px + q = 0$ has a negative discriminant (so that $p < 0$). Vieta made a variable substitution, namely $x = kz$, where $k = 2\sqrt{\frac{P}{3}}$. Under this substitution,

$$x^3 + px + q = 2\left(\frac{P}{3}\right)^{3/2}(4z^3 - 3z + \left(\frac{q}{2}\right)\left(\frac{3}{P}\right)^{3/2}) = 2\left(\frac{P}{3}\right)^{3/2}(4z^3 - 3z - \left(\frac{-q}{2}\right)\left(\frac{3}{P}\right)^{3/2}).$$

(Again, it is important to note that $P = -p$ in the case we are considering.) Now, if $\Delta < 0$, then $-1 < \left(\frac{-q}{2}\right)\left(\frac{3}{P}\right)^{3/2} < 1$. Hence, Vieta made the observation that this quantity must be the cosine of some angle θ. The importance of this observation lies in the fact that, since

$$4\cos^3(\theta) - 3\cos(\theta) - \cos(3\theta) = 0$$

by the sum formula for the cosine, the equation

$$2\left(\frac{P}{3}\right)^{3/2}(4z^3 - 3z - \left(\frac{-q}{2}\right)\left(\frac{3}{P}\right)^{3/2}) = 0$$

has $x = \cos(\theta)$ as a solution. Consequently, $x = k\cos(\theta)$ is a solution of the original cubic equation. Since we must have

$$\cos(3\theta) = \left(\frac{-q}{2}\right)\left(\frac{3}{P}\right)^{3/2},$$

we are able to solve for the angle θ using the inverse cosine. If we do so, we obtain *Vieta's Formula* for solving defective cubics having negative discriminant:

If $x^3 + px + q = 0$ has a negative discriminant, then a solution is given by

$$x = 2\sqrt{\frac{P}{3}}\cos\left[\frac{1}{3}\text{Arccos}\left(\left(\frac{-q}{2}\right)\left(\frac{3}{P}\right)^{3/2}\right)\right].$$

In the case of Example 2, we are dealing with the defective cubic $y^3 - 4y + 1 = 0$. In this case, $P = 4$ and $q = 1$; hence, Vieta's method tells us that

$$y = 2\sqrt{\frac{4}{3}}\cos\left[\frac{1}{3}\text{Arccos}\left(\left(\frac{-1}{2}\right)\left(\frac{3}{4}\right)^{3/2}\right)\right] \approx 1.860805853.$$

A quick check on the calculator shows that this is indeed a solution of the defective cubic equation. Hence, one real root of the polynomial $p(x) = x^3 + 3x^2 - x - 2$ is approximately $x = 2.860805853$.

There is a secondary, rather curious consequence of Vieta's Formula. Since $x_1 = k\cos(\theta)$ is a root of the polynomial $p(z) = 4z^3 - 3z - \cos(3\theta)$, it follows that so also are

$$x_2 = k\cos(\theta + 120^\circ) \text{ and } x_3 = k\cos(\theta + 240^\circ),$$

since $\cos(3(\theta + 120^\circ) = \cos(3\theta) = \cos(3(\theta + 240^\circ)$. Consequently, we have the following result:

If $f(x) = x^3 + px + q$ has a negative discriminant, then f has three real roots.

For the polynomial $p(y) = y^3 - 4y + 1$, the angle θ from Vieta's formula is

$$\theta = \frac{1}{3}\text{Arccos}\left(\left(\frac{-1}{2}\right)\left(\frac{3}{4}\right)^{3/2}\right) \approx 36.3170022^\circ.$$

Therefore, the other two real roots of p are

$$y_2 = k\cos(156.3170022^\circ) \approx -2.114907541 \quad y_3 = k\cos(276.3170022^\circ) \approx .2541016882.$$

Vieta's Formula represented a major breakthrough in the understanding of complex numbers; still, it provides no means of computing complex roots. For this, the world had to wait until the early nineteenth century. Carl Gauss (1777 - 1855), arguably the greatest mathematician of all time, presented the first clear and concise way to understand complex numbers. His approach was to think of complex numbers as vectors on the Cartesian plane.

Let $z = a + bi$ be a complex number. Gauss suggested that we consider z to be a two-dimensional vector whose initial point lay at the origin and whose terminal point lay at the pair (a, b). Using this convention, we can give an alternative characterization for z. Up to a multiple of 2π radians, z is completely determined by magnitude of its vector coupled with the angle θ that it makes with the positive x-axis. The magnitude of z is easily obtained via the Pythagorean Theorem:

$$|\,z\,| = \sqrt{a^2 + b^2}.$$

The magnitude of z is called the *modulus* of z; the angle θ is often called the *argument* of z. Using basic trigonometry, it is easy to see that

$$z = |\,z\,|(\cos(\theta) + i\sin(\theta)).$$

This representation of z is known as the *polar form* of the complex number z. Using this representation for z, it is possible to come up with a concrete motivation for Bombelli's odd (but still used) form of complex multiplication.

To see this motivation, suppose that z_1 and z_2 are complex numbers. For the moment, let us also assume that z_1 and z_2 lie on the unit circle (so that their moduli are both equal to 1). Then, using Gauss's convention and Bombelli's multiplication, we see that

$$\begin{aligned} z_1 z_2 &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2). \end{aligned}$$

Consequently, if z_1 and z_2 lie on the unit circle, Bombelli's scheme for complex multiplication simply corresponds to rotating the vector z_2 through the angle θ_1. The general case is hardly more complicated. If we now assume that $z_1 = |\,z_1\,|(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = |\,z_2\,|(\cos(\theta_2) + i\sin(\theta_2))$, then

$$z_1 z_2 = |\,z_1\,||\,z_2\,|(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Hence, under the vector interpretation for complex numbers, Bombelli's multiplication scheme for complex numbers corresponds to a rotation of z_2 through the angle θ_1 followed by a scaling of z_2 by the modulus of z_1.

Gauss's vector model for complex numbers finally put both the numbers themselves and their unusual multiplication on a firm mathematical footing. Moreover, it is easy to show using mathematical induction that, for any positive integer n, and any complex number $z = |\,z\,|(\cos(\theta) + i\sin(\theta))$, we have

$$z^n = |\,z\,|^n(\cos(n\theta) + i\sin(n\theta)).$$

This result, known as *DeMoivre's Formula* enables us to compute complex roots. To see how, let $z = |\,z\,|(\cos(\theta) + i\sin(\theta)$ and let n be a positive integer. If we let $z^{1/n} = \rho = |\,\rho\,|(\cos(\varphi) + i\sin(\varphi))$ it then follows that

$$|\,z\,|(\cos(\theta) + i\sin(\theta) = \rho^n = |\,\rho\,|^n(\cos(n\varphi) + i\sin(n\varphi)).$$

Equating coefficients, we see that $|\,z^{1/n}\,| = |\,z\,|^{1/n}$. We also see that $n\varphi = \theta$ (up to a multiple of 2π). This tells us that we can take

$$\varphi = \frac{\theta + 2j\pi}{n},$$

where j is any integer such that $0 \le j < n$. Consequently, we are left with the rather surprising fact that a complex number always has exactly n complex nth roots. Moreover, if we let $z = |\,z\,|(\cos(\theta) + i\sin(\theta))$, then these roots take the form

$$\sqrt[n]{z} = \sqrt[n]{|\,z\,|}\left[\cos\left(\frac{\theta + 2\pi j}{n}\right) + i\sin\left(\frac{\theta + 2\pi j}{n}\right)\right].$$

As an example, let us return to the equation Bombelli considered:$x^3 - 15x - 4 = 0$. We know that Cardano's Formula gives us

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}} = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Consider $\rho = \sqrt[3]{2 + \sqrt{-121}} = \sqrt[3]{2 + 11i}$. In this case, we have $z = 2 + 11i$; hence, we know that $|\,z\,| = \sqrt{125} = 5\sqrt{5}$. Consequently, the polar form for z is

$$z = 5\sqrt{5}\left(\frac{2\sqrt{5}}{25} + \frac{11\sqrt{5}}{25}\,i\right).$$

If we let θ denote the argument of z, then we know $\cos(\theta) = 2\sqrt{5}/25$ and $\sin(\theta) = 11\sqrt{5}/25$. Consequently, we know

$$\theta = \text{Arcsin}\left(\frac{11\sqrt{5}}{25}\right) \approx 1.39094\text{radians}.$$

Therefore, the three complex cube roots of $z = 2 + 11i$ are

(1) $\rho_0 = 2.236068\left[\cos\left(\frac{1.39094}{3}\right) + i\sin\left(\frac{1.39094}{3}\right)\right] \approx 2.236068(.8944276 + .4472127\,i) = 2 + i$

(2) $\rho_1 = 2.236068\left[\cos\left(\frac{1.39094 + 2\pi}{3}\right) + i\sin\left(\frac{1.39094 + 2\pi}{3}\right)\right] \approx 2.236068(-.8345114 + .5509907\,i)$

(3) $\rho_2 = 2.236068\left[\cos\left(\frac{1.39094 + 4\pi}{3}\right) + i\sin\left(\frac{1.39094 + 4\pi}{3}\right)\right] \approx 2.236068(-.0599162 - .9982034\,i).$

Notice that Bombelli's solution appears as one of the roots. Of course, we can also compute the three complex cube roots of $z' = 2 - 11i$:

(1) $\rho_0' = 2.236068\left[\cos\left(\frac{1.39094}{3}\right) - i\sin\left(\frac{1.39094}{3}\right)\right] \approx 2.236068(.8944276 - .4472127\,i) = 2 - i$

(2) $\rho_1' = 2.236068 \left[\cos\left(\frac{1.39094 + 2\pi}{3}\right) - i\sin\left(\frac{1.39094 + 2\pi}{3}\right)\right] \approx 2.236068(-.8345114 - .5509907\,i)$

(3) $\rho_2' = 2.236068 \left[\cos\left(\frac{1.39094 + 4\pi}{3}\right) - i\sin\left(\frac{1.39094 + 4\pi}{3}\right)\right] \approx 2.236068(-.0599162 + .9982034\,i)$.

Adding these complex roots together gives us the three correct roots for the original cubic polynomial.

Part III: The Quartic Equation

Having examined the solution of the cubic equation in some depth, we now turn briefly to the quartic equation. Traditional approaches to solving the quartic equation all involve reducing the problem to some collection of quadratics and cubics which can be attacked using Cardano's methods. All methods are difficult (at best) to use. The earliest solution seems to be due to Ludovico Ferrari (1522-1565), one of Cardano's students. Vieta later contributed another, more efficient solution. We begin by considering the general quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

where a, b, c, d are arbitrary real numbers. As with the cubic, it is possible to reduce this equation to a "defective" one in which the third-degree term is absent. In this case, the substitution $x = y - a/4$ does the trick, reducing the general quartic equation to

$$y^4 + py^2 + qy + r = 0,$$

where $p = b - \dfrac{3a^2}{2}, q = c + \dfrac{a^3 - ba}{2}, r = d - \dfrac{3a^4 + 64ac}{256}$.

To understand Ferrari's method, we consider a specific problem: Solve the equation $y^4 - 10y^2 + 4y + 8 = 0$. Ferrari would have begun by rewriting this equation as

$$y^4 - 10y^2 = -4y - 8$$

keeping the even powers on one side. Ferrari then decided to reduce the left-hand side to $(y^2 - 10)^2$, since this is easy to solve for y. In order to accomplish this simplification, he added $100 - 10y^2$ to both sides, obtaining

$$(y^2 - 10)^2 = -4y - 8 + (100 - 10y^2) = -10y^2 - 4y + 92$$

The problem would be solved if the right-hand side can also be converted into a perfect square. In order to do this without affecting the perfect square on the left, Ferrari decided to add a new unknown z into the left-hand term. Of course, the unknown would also have to added into right expression to preserve equality; Ferrari hoped to be able to determine a value for the unknown which would force the right-hand side to become a perfect square.

If we expand $(y^2 - 10 + z)^2$, we obtain $y^4 - 20y^2 + 2y^2z - 20z + z^2 + 100$. In order to preserve equality in the original equation, this told Ferrari that he must add the missing terms of this expansion to the right-hand side. In particular, he obtained

$$(y^2 - 10 + z)^2 = -10y^2 - 4y + 92 + 2z(y^2 - 10) + z^2 = (2z - 10)y^2 - 4y + (92 - 20z + z^2)$$

The problem he now faced was determining which value (if any) of z would force the right-hand side to be a perfect square. He observed that the right-hand expression is a quadratic in y of the form $\alpha y^2 + \beta y + \gamma$. This quadratic will be a perfect square if and only if its discriminant is zero; that is, if and only if $\beta^2 - 4\alpha\gamma = 0$. Ferrari used this fact with $\beta = -4$, $\alpha = 2z - 10$, and $\gamma = 92 - 20z + z^2$ to obtain the condition

$$16 = 4(2z - 10)(92 - 20z + z^2) = z^3 - 25z^2 + 192z - 446.$$

This, of course, led to the cubic equation $z^3 - 25z^2 + 192z - 462 = 0$. Ferrari had "solved" the problem in the sense that he had reduced it to a problem involving a cubic and perfect squares.

Of course, there was still considerable work to be done, but there was nothing new to determine; the means of solving the cubic was already known. Using the substitution $z = u + 25/3$, the cubic in z reduces to

$$u^3 - \frac{49}{3}u - \frac{524}{27} = 0.$$

Invoking Cardano's formula, one finds that $u = -4/3$ is a solution; hence, $z = 7$ will make the expression $(2z - 10)y^2 - 4y + (92 - 20z + z^2)$ a perfect square. Indeed,

$$(2(7) - 10)y^2 - 4y + (92 - 20(7) + 49) = 4y^2 - 4y + 1 = (2y - 1)^2.$$

Consequently, the original equation $y^4 - 10y^2 + 4y + 8 = 0$ becomes

$$(y^2 - 3)^2 = (2y - 1)^2.$$

Solving this equation gives four answers, namely $y = 1 \pm \sqrt{3}$ and $y = -1 \pm \sqrt{5}$.

Any defective quartic equation may be solved using the steps outlined above. The key is to take $y^4 + py^2 + qy + r = 0$ and rewrite it in the form $y^4 + py^2 = -qy - r$. You then proceed to make the left-hand side a perfect square by adding $py^2 + p^2$ to both sides. Next, you introduce an unknown z into the equation geared to force the right-hand side to be a perfect square as well, ending up with the equation

$$(y^2 + p + z)^2 = (p + 2z)y^2 - qy + (p^2 - r + 2pz + z^2).$$

The right-hand side will be a perfect square in y if and only if z satisfies the equation

$$q^2 = 4(p + 2z)(p^2 - r + 2pz + z^2) = 8z^3 + 20pz^2 + (16p^2 - 8r)z + (4p^3 - 4pr).$$

This is a cubic in z, which is solved using Cardano's formula.