MATH 1920 PRACTICE EXAM I

100 points

NAME:____________________________

Please place the letter of your selection in the blank provided. These questions are worth five points each.

5 pts 1. __**B**__ One antiderivative for $f(x) = e^x$ is the function

(a) $F(x) = \frac{1}{2}e^{x^2}$ (b) $F(x) = e^x - \sqrt{2}$

(c) $F(x) = \frac{1}{2}e^{2x}$ (d) $F(x) = \ln(x) + \pi$

(e) $F(x) = 3e^x$

Justification. We need a function whose derivative is f.

5 pts 2. __**B**__ In order to compute the antiderivative family for $f(x) = \frac{\cos(1/x)}{x^2}$ we need the substitution

(a) $u = \cos(x)$ (b) $u = \frac{1}{x}$

(c) $u = \frac{1}{x^2}$ (d) $u = x^2$

(e) $u = x$

Justification. The general rule for substitution is to let u be the argument of the composite factor. Of course, any substitution is valid only if its derivative also appears as a factor the function f.

5 pts 3. __**A**__ The function $F(x) = x\sin(x) + 10$ is an antiderivative for

(a) $f(x) = x\cos(x) + \sin(x)$ (b) $f(x) = \frac{x^2}{2}\sin(x) - \cos(x)$

(c) $f(x) = -\frac{x^2}{2}\cos(x)$ (d) $f(x) = x\cos(x)$

(e) $f(x) = \frac{x^2}{2}\cos(x)$

Justification. Take the derivative of the function F.

In Problem 4, suppose we know that

$$\int_a^b f(x)dx = -5 \qquad \int_a^b g(x)dx = 10$$

5 pts 4. __**E**__ Based on this information, we know $\int_a^b [3f(x) - 6g(x)]\,dx$

(a) is equal to 75 (b) is equal to -5

(c) is equal to 5 (d) is equal to -45

(e) is equal to -75

Justification. Apply the Sum and Constant Multiple Rules.

$$\int_a^b [3f(x) - 6g(x)]\,dx = 3\int_a^b f(x)dx - 6\int_a^b g(x)dx = -75$$

5 pts 5. ___E___ By making an appropriate substitution, we know that $\int \frac{x^2}{(1+x^3)^4} dx$

(a) is equal to $3\int \frac{u}{(1+u)} du$ (b) is equal to $\int x^2 \left(\frac{1}{u^4}\right) du$

(c) is equal to $\frac{1}{2}\int \frac{u}{1+u^6} du$ (d) is equal to $\int u^{-4} du$

(e) is equal to $\frac{1}{3}\int u^{-4} du$

5 pts 6. ___C___ After applying the appropriate substitution in $\int_{x=1}^{x=4} \frac{\sin(2+\ln(x))}{x} dx$, the new limits will be

(a) $u = 1$ and 4 (b) $u = 1$ and $u = 1/4$
(c) $u = 2$ and $u = 2 + \ln(4)$ (d) $u = \sin 2$ and $u = \sin(2 + \ln(4))$
(e) $u = 1$ and $u = 2$

Justification. The correct substitution is $u = 2 + \ln(x)$, since $\frac{du}{dx} = \frac{1}{x}$ is a factor of the integrand.

5 pts 7. ___E___ If $F(t) = \int_1^t x\sqrt{x^2-1} dx$, then $F'(3)$

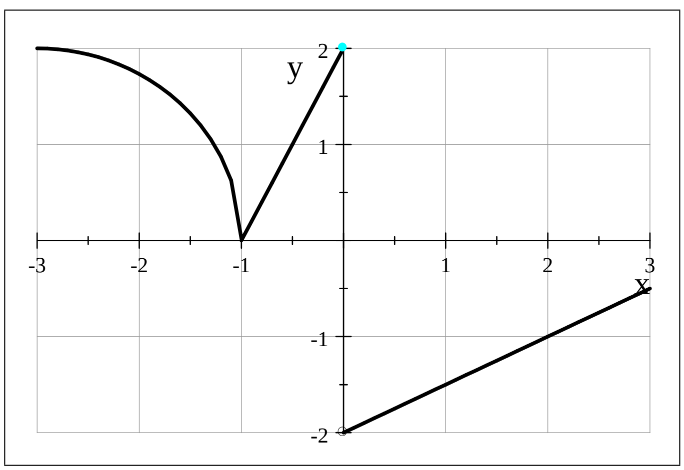
(a) is equal to $\sqrt{3}$ (b) is equal to $2\sqrt{2}$

(c) is equal to $\frac{2\sqrt{2}}{3}$ (d) is equal to $\frac{2\sqrt{3}}{3}$

(e) is equal to $6\sqrt{2}$

Justification. The First Fundamental Theorem of Calculus tells us that $F'(t) = t\sqrt{t^2-1}$.

Problems 8 - 10 refer to the graph of a function f below. The curve is an arc of a circle of radius 2.



5 pts 8. ___D___ Based on this diagram, we know $\int_{-3}^{2} f(x) dx$

(a) is equal to $\frac{4\pi - 7}{4}$ (b) is equal to $\pi - 1$

(c) is equal to $\frac{\pi}{4} - 3$ (d) is equal to $\pi - 2$

(e) is equal to $3 - \frac{\pi}{4}$

Justification. The net area in question consists of a quarter circle of radius 2 and right triangle above the x-axis and a trapezoid below the x-axis. Consequently, we know

$$\int_{-3}^{2} f(x)dx = \int_{-3}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx + \int_{0}^{2} f(x)dx = \frac{1}{4}(4\pi) + 1 - 3$$

5 pts 9. **E** If $F(x) = \int_0^x f(t)dt$, then $F(-1)$

(a) is equal to 1 (b) is equal to 0

(c) is equal to $\frac{1}{2}$ (d) is equal to $-\frac{1}{2}$

(e) is equal to -1

Justification. We know that

$$F(-1) = \int_0^{-1} f(t)dt = -\int_{-1}^{0} f(t)dt = -1$$

5 pts 10. **E** The arc-length of the function $f(x) = \cos^2(x)$ on the interval $0 \leq x \leq \pi$ is given by the formula

(a) $\int_0^{\pi} \cos(x)dx$ (b) $\int_0^{\pi} \cos^2(x)dx$

(c) $\int_0^{\pi} (1 + 2\sin(x)\cos(x))\, dx$ (d) $\int_0^{\pi} \sqrt{1 + 2\sin(x)\cos(x)}dx$

(e) $\int_0^{\pi} \sqrt{1 + 4\sin^2(x)\cos^2(x)}dx$

Justification. Arc-length is given by the formula $\mathcal{L} = \int_a^b \sqrt{1 + [f'(x)]^2}dx$.

10 pts 11. What is the average value of the function $f(x) = x - \frac{1}{x}$ on the interval $1 \leq x \leq 6$? Show your work.

Solution. The average value is given by the formula

$$\begin{aligned} A &= \frac{1}{6-1}\int_1^6 \left(x - \frac{1}{x}\right) dx \\ &= \frac{1}{5}\left(\frac{x^2}{2} - \ln|x|\Big|_1^6\right) \\ &= \frac{1}{5}\left[\left(\frac{36}{2} - \ln(6)\right) - \left(\frac{1}{2} - \ln(1)\right)\right] \\ &= \frac{1}{5}\left(\frac{35}{2} - \ln(6)\right) \end{aligned}$$

10 pts 12. Use the Fundamental Theorems of Calculus to compute $\int_0^1 \frac{x^2}{(1+x^3)^4}dx$ . . Show your work.

Solution. The exact value will be given by

$$\begin{aligned}
\int_0^1 \frac{x^2}{(1+x^3)^4}dx &= \int_0^1 \frac{x^2}{(1+x^3)^4}dx \qquad \text{Let } u = 1+x^3 \text{ so } x^2 = \frac{1}{3}\frac{du}{dx} \\
&= \int_{u=1}^{u=2} \frac{1}{u^4}\left[\frac{1}{3}\frac{du}{dx}\right]dx \\
&= \frac{1}{3}\int_{u=1}^{u=2} u^{-4}du \\
&= -\frac{1}{9}u^{-3}\Big|_{u=1}^{u=2} \\
&= -\frac{1}{9}\left(\frac{1}{8}-1\right) \\
&= \frac{7}{56}
\end{aligned}$$

15 pts 13. Consider the function $f(x) = \sqrt{1+4x^2}$ on the interval $[1,3]$.

(a) If we divide $[1,3]$ into six subintervals of equal width, then the partition we obtain is

$x_0 =$ 1 $\quad x_1 =$ 4/3 $\quad x_2 =$ 5/3 $\quad x_3 =$ 2

$x_4 =$ 7/3 $\quad x_5 =$ 8/3 $\quad x_6 =$ 3

(b) Use a trapezoid estimate with six subintervals to estimate $\int_1^3 f(x)dx$. Show your work.

Solution. Observe that $\triangle x = 1/3$. With this in mind, the left-hand estimate will be

$$\begin{aligned}
\int_1^3 f(x)dx &\approx \frac{1}{6}\left(\sqrt{1+4[1]^2} + 2\sqrt{1+4\left[\frac{4}{3}\right]^2} + 2\sqrt{1+4\left[\frac{5}{3}\right]^2} + 2\sqrt{1+4\,[2]^2}\right) + \\
&\quad \frac{1}{6}\left(2\sqrt{1+4\left[\frac{7}{3}\right]^2} + 2\sqrt{1+4\left[\frac{8}{3}\right]^2} + \sqrt{1+4(3)^2}\right) \\
&\approx \frac{1}{6}(2.236 + 5.696 + 6.96 + 8.246 + 9.546 + 10.852 + 6.083) \\
&\approx 8.27
\end{aligned}$$

10 pts 14. Use the Fundamental Theorems of Calculus to compute the exact value of $\int_0^1 \left(2+\sin(\pi x) - \frac{x}{1+x^2}\right)dx$. Show your work.

Solution. Observe that

$$\begin{aligned}
\int_0^1 \left(2+\sin(\pi x) - \frac{x}{1+x^2}\right)dx &= \int_0^1 2dx + \int_0^1 \sin(\pi x)dx - \int_0^1 \frac{x}{1+x^2}dx \\
&= 2x\big|_{x=0}^{x=1} + \frac{1}{\pi}\int_{u=0}^{u=\pi}\sin(u)du - \int_0^1 \frac{x}{1+x^2}dx \qquad \text{Let } u = \pi x \text{ so } \frac{du}{dx} = \pi. \\
&= 2x\big|_{x=0}^{x=1} - \frac{1}{\pi}\cos(u)\big|_{u=0}^{u=\pi} - \frac{1}{2}\int_{u=1}^{u=2}\frac{1}{u}du \qquad \text{Let } u = 1+x^2 \text{ so } \frac{du}{dx} = 2x. \\
&= 2x\big|_{x=0}^{x=1} - \frac{1}{\pi}\cos(u)\big|_{u=0}^{u=\pi} - \frac{1}{2}\ln|u|\big|_{u=1}^{u=2} \\
&= [2(1)-2(0)] - \frac{1}{\pi}[\cos(\pi)-\cos(0)] - [\ln(2)-\ln(1)] \\
&= 2 + \frac{2}{\pi} - \ln(2)
\end{aligned}$$