INTRODUCTION TO INTEGRATION

1 Part 1 — Net Area

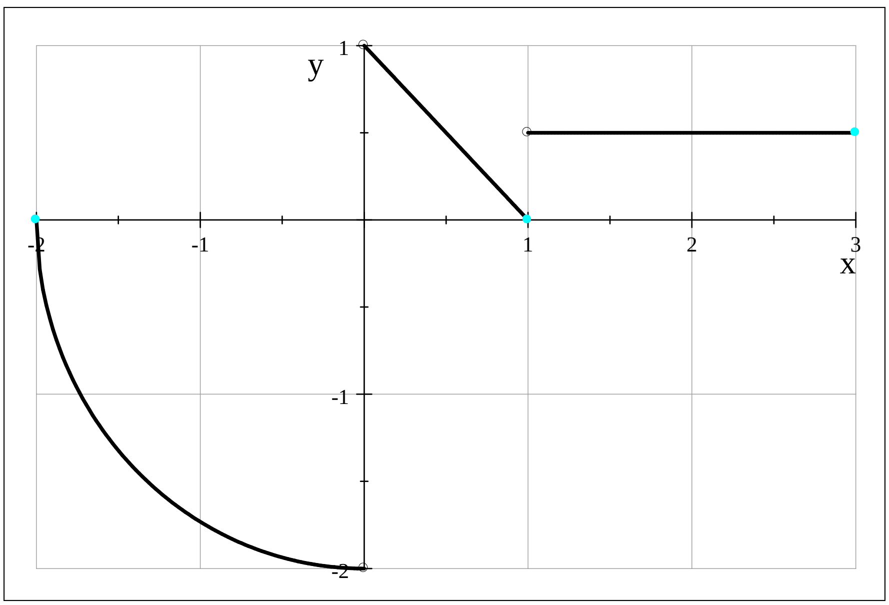
A function f is said to be *integrable* on a closed interval $a \leq x \leq b$ provided the following conditions are met

- The function f is defined for all but a finite (possibly empty) set of numbers in $[a, b]$.
- The function f has at most finitely many jump discontinuities on $[a, b]$.
- The function f has no vertical asymptotes on $[a, b]$.

For any function f that is integrable on a closed interval $[a, b]$, we define the *net area* for f on this interval to be

$$\int_a^b f(x)dx = \text{[Area below graph of } f \text{ and above } x\text{-axis]} - \text{[Area above graph of } f \text{ and below } x\text{-axis]}$$

For example, consider the function f whose graph is shown below.



The function f is integrable on the interval $[-2, 3]$. Assuming that the curved portion of the graph is part of a circular arc of radius 2, we can compute the net area for f on this interval. The area above the graph of f and below the x-axis is the area of the quarter circle. In particular, we know

$$\text{Area above graph of } f \text{ and below } x\text{-axis} = \frac{1}{4}(4\pi) = \pi$$

The area below the graph of f and above the x-axis is determined by two regions — a triangle and a rectangle. In particular, we know

$$\begin{aligned}
\text{Area below graph of } f \text{ and above } x\text{-axis} &= \text{[Area of Triangle]} + \text{[Area of Rectangle]} \\
&= \left[\frac{1}{2}\text{Base} \times \text{Height}\right] + [\text{Base} \times \text{Height}] \\
&= \left[\frac{1}{2}(1)(1)\right] + \left[(2)\left(\frac{1}{2}\right)\right] \\
&= 1.5
\end{aligned}$$

Consequently, the net area for f is given by

$$\int_{-2}^{3} f(x)dx = 1.5 - \pi \approx -1.64159$$

Notice that, unlike true area, the net area for a function can be negative. A negative net area simply means there is more area below the x-axis than above the x-axis. If the graph of an integrable function f is entirely above the x-axis on an interval $[a, b]$, then

$$\int_{a}^{b} f(x)dx$$

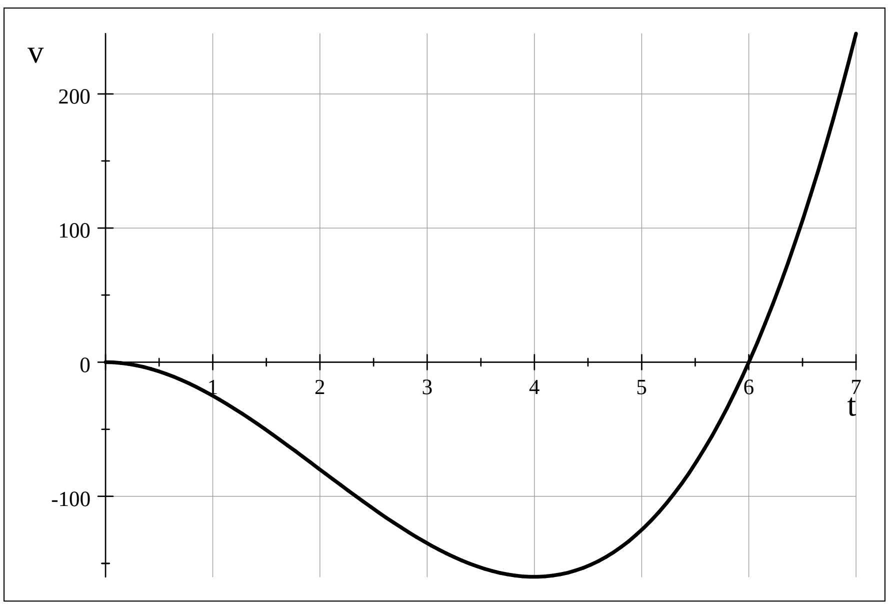
gives the true area between the graph of f and the x-axis on this interval.

Problem 1. Using the diagram above, compute the following net areas.

$$\int_{-2}^{0} f(x)dx \qquad \int_{1/2}^{2} f(x)dx \qquad \int_{-2}^{3/2} f(x)dx$$

As long as the graph of a function f is composed of arcs from circles or straight line segments on a closed interval, then it is a simple matter to compute the net area for f on that interval. It is more challenging, however, if the graph of f is more complicated.

Example 1 *Young Doris is riding her tricycle back and forth along a straight road leading away from a lake. The graph below shows her velocity $v = f(t)$ as a function of time. Negative velocities represent travel toward the lake, and positive velocities represent travel away from the lake. Time is measured in minutes, and velocity is measured in feet per minute. What is the approximate net distance along the road that Doris tavels between $t = 5$ and $t = 7$ minutes?*



Solution. At first, it may not seem possible to answer this question based on the information we have. However, take a closer look at the "area" of each rectangle on the diagram. The base of each rectangle is measured in minutes, while the height of each rectangle is measured in feet traveled per minute. Consequently, the units of "area" for each rectangle is actually *feet traveled.* Consequently, the net area for the velocity function between two times represents the distance traveled between these times. In other words,

$$\int_{5}^{7} f(t)dt = \text{ Net distance Doris travels between } t = 5 \text{ and } t = 7 \text{ minutes}$$

Now, the velocity curve on this interval is certainly not composed of simple geometric shapes, so it is not possible to use elementary area formulas to compute the net area exactly. However, we can *approximate* the net area in a relatively simple way.

1. First, take the interval $[5, 7]$ and divide it into a number of subintervals of equal width.
2. Construct a rectangle on each subinterval by choosing a point in each subinterval and letting the signed height of the rectangle be the function value at this point.
3. Add up the signed areas of these rectangles.

The more subintervals you use, the better an approximation to the net area this sum will be. Suppose we decide to break the interval $[5, 7]$ into ten subintervals of equal width. Let's call these subintervals I_0, I_1, ..., I_9. Since all have equal width, we know the width of each one will be

$$\Delta t = \frac{7-5}{10} = .2$$

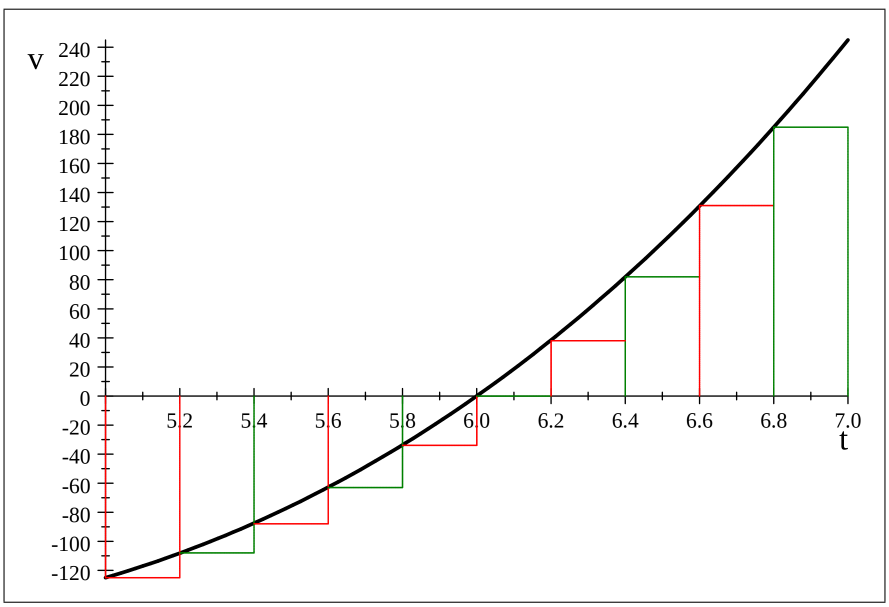
This means that each subinterval is

$$I_0 = [5, 5.2] \qquad I_1 = [5.2, 5.4] \qquad I_2 = [5.4, 5.6] \qquad I_3 = [5.6, 5.8]$$

$$I_4 = [5.8, 6.0] \qquad I_5 = [6.0, 6.2] \qquad I_6 = [6.2, 6.4] \qquad I_7 = [6.4, 6.6]$$

$$I_8 = [6.6, 6.8] \qquad I_9 = [6.8, 7.0]$$

Now that we have created our subintervals, we create rectangles by selecting a number from each one and evaluating the velocity function at these numbers. We are free to choose these numbers any way that we want. To be consistent, let's select the left endpoint from each subinterval. To estimate the function values, we use the graph. The diagram below shows a closeup of Doris's velocity function on the interval $[5, 7]$



Interval Number	0	1	2	3	4	5	6	7	8	9
Left Endpoint of Subinterval	5.0	5.2	5.4	5.6	5.8	6.0	6.2	6.4	6.6	6.8
Approximate Output of f at Endpoint	-125	-108	-88	-63	-34	0	38	82	131	185
Signed Area of Rectangle	-25	-21.6	-17.6	-12.6	-6.8	0	7.6	16.4	26.2	37

Based on these estimated values, Doris' net distance traveled is approximated by the sum of the signed areas of all these rectangles.

$$\int_5^7 f(x)dx \approx -25 - 21.6 - 17.6 - 12.6 - 6.8 + 0 + 7.6 + 16.4 + 26.2 + 37 = 3.6 \text{ feet}$$

Of course, this is not Doris' *actual* distance traveled; it is only her *net* distance traveled. The fact that it is rather small indicates that she traveled about as far away from the lake as she traveled toward the lake.

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Problem 2. The digram below shows a closeup of Doris' velocity function on the time interval $[0, 3]$.

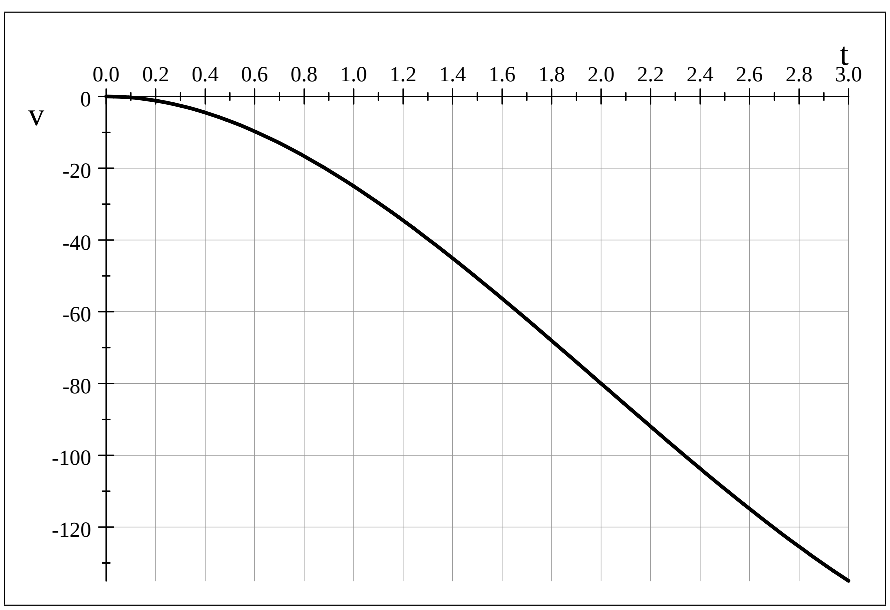
Part (a): Divude this time interval into six equal parts. How wide will each subinterval be?

Part (b): Using the left endpoint of each subinterval, draw the approximating rectangle whose signed height is the output of f at this endpoint.

Part (c): Use the graph to estimate the signed height of each approximating rectangle and fill in the table.

Interval Number	0	1	2	3	4	5
Left Endpoint of Subinterval						
Approximate Output of f at Endpoint						
Signed Area of Rectangle						

Part (d): Use the signed areas of these approximating rectangles to estimate $\int_0^3 f(x)dx$.



In the previous example, we could have obtained a better approximation for Doris' net distance traveled if we had divided the interval $[5, 7]$ into even smaller subintervals. The smaller the subintervals, the better each rectangle will fit under the curve and the less error there will be in the estimation of the net area. This suggests that we can think of the net area as a kind of *limit process*.

Suppose that f is an integrable function on a closed interval $[a, b]$. We can divide the interval into n subintervals of equal width by letting

$$\triangle x = \frac{b-a}{n}$$

and selecting a sequence of numbers

$$x_0 = a \qquad x_1 = a + \triangle x \qquad x_2 = a + 2\triangle x \qquad ... \qquad x_{n-1} = a + (n-1)\triangle x \qquad x_n = b$$

In this case, each subinterval has consecutive numbers in this sequence as its endpoints. In particular, for all integers j such that $0 \leq j \leq n-1$, we have

$$I_j = [x_j, x_{j+1}]$$

Now, by consistently evaluating the function f at the left endpoint of each subinterval, we create a sequence of n rectangles, and the net area for f is approximated by the sum of the signed areas of all these rectangles:

$$\int_a^b f(x)dx \approx f(x_0)\triangle x + f(x_1)\triangle x + ... + f(x_{n-1})\triangle x = \sum_{j=0}^{n-1} f(x_j)\triangle x$$

This approximation to the net area is called a *left hand* approximation, since it uses the left endpoint of each subinterval to form the signed heights of the rectangles. We used a left hand approximation in the previous example.

Example 2 *Using seven subintervals of equal width, construct a left hand approximation for*

$$\int_1^2 \sqrt{1+x^2}dx$$

Solution. We first divide the interval $[1, 2]$ into seven subintervals of equal width. In this case,

$$\triangle x = \frac{2-1}{7} = \frac{1}{7}$$

Using this width, we construct the bases of our approximating rectangles using the following sequence of numbers:

$$x_0 = 1 \qquad x_1 = 1 + \frac{1}{7} = \frac{8}{7} \qquad x_2 = 1 + \frac{2}{7} = \frac{9}{7} \qquad x_3 = 1 + \frac{3}{7} = \frac{10}{7}$$

$$x_4 = 1 + \frac{4}{7} = \frac{11}{7} \qquad x_5 = 1 + \frac{5}{7} = \frac{12}{7} \qquad x_6 = 1 + \frac{6}{7} = \frac{13}{7} \qquad x_7 = 2$$

The signed height of each rectangle is obtained by evaluating the function $f(x) = \sqrt{1+x^2}$ at the left endpoint of each subinterval. In practical terms, this means we will use the first seven numbers in the sequence (which means we omit x_7) . We have

$$\begin{aligned}
\int_1^2 \sqrt{1+x^2}dx &\approx \sum_{j=0}^{6} \sqrt{1+(x_j)^2}\triangle x \\
&= \sqrt{1+(1)^2}\left(\frac{1}{7}\right) + \sqrt{1+\left(\frac{8}{7}\right)^2}\left(\frac{1}{7}\right) + \sqrt{1+\left(\frac{9}{7}\right)^2}\left(\frac{1}{7}\right) + \sqrt{1+\left(\frac{10}{7}\right)^2}\left(\frac{1}{7}\right) \\
&\quad + \sqrt{1+\left(\frac{11}{7}\right)^2}\left(\frac{1}{7}\right) + \sqrt{1+\left(\frac{12}{7}\right)^2}\left(\frac{1}{7}\right) + \sqrt{1+\left(\frac{13}{7}\right)^2}\left(\frac{1}{7}\right) \\
&\approx \left(\frac{1}{7}\right)[1.414 + 1.519 + 1.629 + 1.744 + 1.863 + 1.985 + 2.109] \\
&\approx 1.752
\end{aligned}$$

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How accurate is the estimate we obtained in the previous example? At the moment, we have no way of knowing directly. We could use more subintervals (say 10 or 20 subintervals) and compare calculations. We could also try creating our rectangles using different points. For example, instead of computing the signed heights of our approximating rectangles by using the left endpoint of each subinterval, we could use the right endpoint, or even the midpoint instead.

Problem 3. Using seven subintervals of equal width, the the right endpoint of each subinterval to construct a *right* hand approximation for

$$\int_1^2 \sqrt{1+x^2}dx$$

There is a fairly substantial difference between the left hand and the right hand estimates above. This is typical when we use a relatively small number of subintervals. These estimates grow closer together as the number of subintervals increases. If we had used twenty subintervals instead of seven, for example, the left and right hand estimates would be much closer together.

One way to reduce the likely error in our approximations would be to take the left and right hand estimates for a given number of subintervals and average them together. If we took the average of the left and right hand approximations in the previous two examples, we would have

$$\int_1^2 \sqrt{1+x^2}dx \approx \frac{1}{2}(1.752+1.869) = 1.8105$$

We could be fairly confident that this estimate is better than either the left or the right hand approximation by itself. We still, however, do not know how close this estimate is to the actual value.

This process of averaging the left and right hand estimates for net area can be rewritten as an estimating formula in its own right:

- If f is an integrable function on an interval $[a,b]$, then the *trapezoid* estimate of the net area for f on

$[a, b]$ using n subintervals of equal width is given by

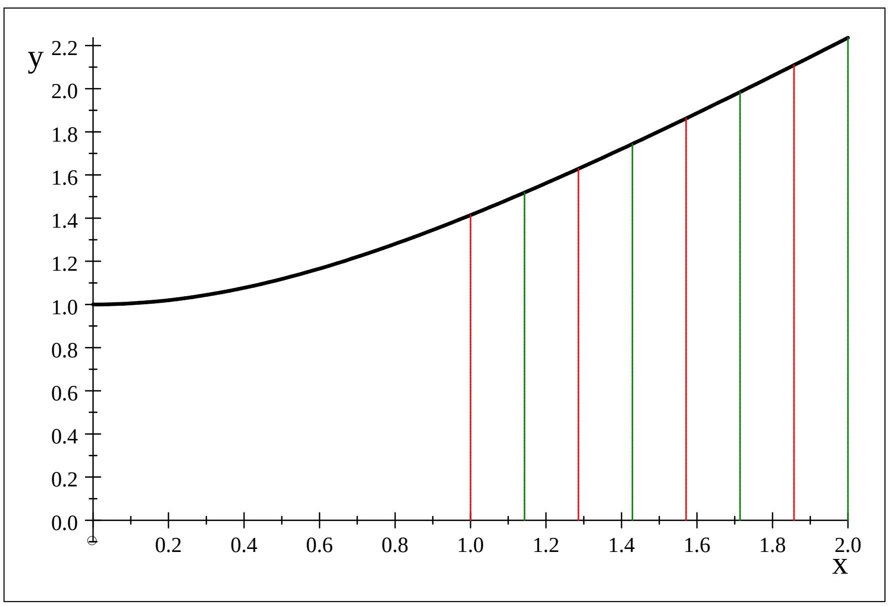
$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{1}{2}\left(\sum_{j=0}^{n-1} f(x_j)\triangle x + \sum_{j=0}^{n-1} f(x_{j+1})\triangle x\right) \\ &= \sum_{j=0}^{n-1} \frac{[f(x_j) + f(x_{j+1})]}{2}\triangle x \\ &= \frac{\triangle x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)]\end{aligned}$$

The average of the left and right hand approximations is called the trapezoid estimate because the formula

$$\frac{f(x_j) + f(x_{j+1})}{2}\triangle x$$

happens to be the area of the trapezoid whose base is the interval $[x_j, x_{j+1}]$ and whose heights are $f(x_j)$ and $f(x_{j+1})$ when both of these values are positive.

The diagram below shows the seven approximating trapezoids for $f(x) = \sqrt{1 + x^2}$ on the interval $[1, 2]$. Notice how well the trapezoids fit under the curve — there is virtually no error at all. Consequently, we would expect that the trapezoid estimate for the net area would be very accurate.



EXERCISES FOR PART 1

Problems 1-6 refer to the graph of Doris' velocity function presented in Example 1.

1. Use a right hand approximation with four subintervals of equal width to estimate the net distance Doris traveled between $t = 1$ and $t = 3$ minutes.
2. Use a trapezoid approximation with eight subintervals of equal width to estimate the net distance Doris traveled between $t = 1$ and $t = 3$ minutes.
3. Could Doris have started her tricycle ride at the lakeshore? Explain your answer.
4. At approximately what time was Doris closest to the lakeshore? Explain your answer.

5. Suppose that Doris started her journey 600 feet from the lakeshore. Use a trapezoid estimate with twelve subintervals to approximate how far Doris was from the lakeshore at her closest point.

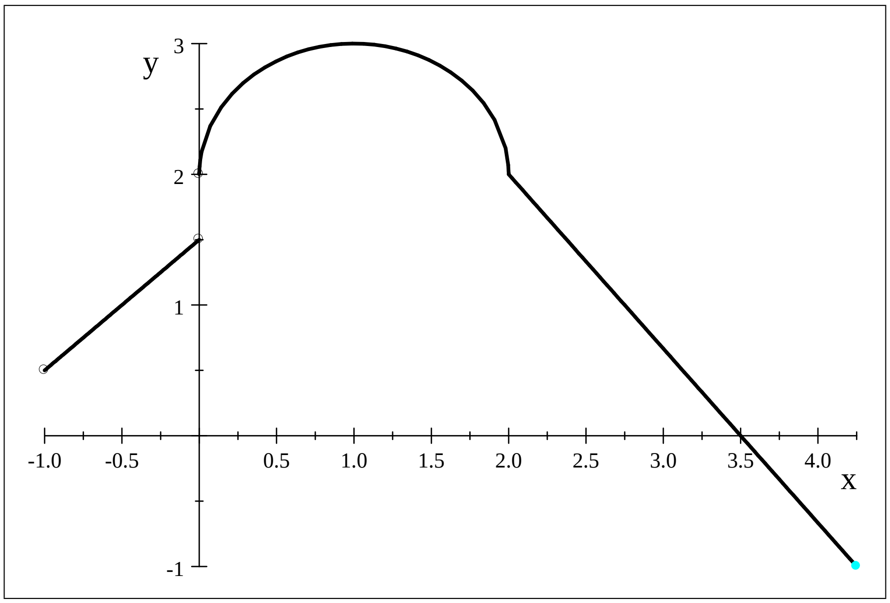
6. Use a left hand estimate with six subintervals to approximate $\int_0^{3/2} \cos(\pi x)dx$.

7. Use a trapezoid estimate with eight subintervals to approximate $\int_1^3 \ln(1+x^2)dx$.

8. Use a right hand estimate with five subintervals to approximate $\int_{-1}^{1} (3x-1)^3\, dx$.

9. Use a left hand estimate with seven subintervals to approximate $\int_{-2}^{0} \frac{1}{x^2+1}dx$.

   Problems 10-12 refer to the graph of f shown below. The curved portion of the graph is a semicircle of radius 1 centered at $(1,2)$.



10. Find the exact value of $\int_{-1}^{1} f(x)dx$.

11. Find the exact value of $\int_0^{4.25} f(x)dx$.

12. Find the exact value of $\int_1^{3.5} f(x)dx$.

2 Part 2 — The Definite Integral

In the previous section, we introduced the notion of net area between the graph of a function f and the x-axis on a closed interval $[a,b]$. We also provided some methods for approximating this net area when the graph is not composed of line segments or arcs of circles. In this section, we lay some of the groundwork necessary to compute these net areas exactly.

It stands to reason that the left hand, right hand, or trapezoid approximations for net area get better and better as the number n of subintervals increases. With this in mind, we make the following definition

Definition 3 *Let f be an integrable function on the closed interval between the numbers $x = a$ and $x = b$. The definite integral for f on this interval is defined to be*

$$\int_a^b f(x)dx = \lim_{n \longrightarrow \infty} \sum_{j=0}^{n-1} f(x_j^*)\triangle x$$

where $\triangle x = (b-a)/n$, $x_j = a + j\triangle x$ for $0 \leq j \leq n$, and x_j^* is any number chosen from the subinterval $I_j = [x_j, x_{j+1}]$.

In Part 1, we always let the number x_j^* be either the left or the right endpoint of the subintervals. However, it really does not matter what number you choose to compute the signed heights of the rectangles on the subintervals. Individual approximations can vary considerably based on the choice of these numbers, but *in the limit*, all become equal. (This is not easy to prove; we will simply assume it in this course.) Expressions of the type

$$\sum_{j=0}^{n-1} f(x_j^*)\triangle x$$

are called *Riemann sums* (named in honor of Bernhard Riemann), and the definition above is often called the *Riemann* definition of the definite integral.

The Riemann definition of the definite integral is seldom used for actual computation, since the limit itself is difficult to interpret. However, the Riemann sum definition is critical to developing applications of the definite integral. Here is one example.

We now consider a somewhat different example, one which will be important in the next section. Suppose that f is a continuous function on a closed interval $[a, b]$ and suppose further that we have selected n equally spaced numbers $x_0 = a$, ... $x_{n-1} = b$ from this interval. If we form the average (or *arithmetic mean*) of the numbers $f(x_j)$, observe that we obtain

$$\text{Ave}_n = \frac{f(x_0) + f(x_1) + ... + f(x_{n-1})}{n} = \frac{f(x_0) + f(x_1) + ... + f(x_{n-1})}{b-a}\left(\frac{b-a}{n}\right) = \frac{1}{b-a}\sum_{j=0}^{n-1} f(x_j)\triangle x$$

In other words, the average of these n function values is actually a Riemann sum. If we take the limit of these averages as n approaches infinity, we therefore obtain a definite integral known as the *average value* of the function f on the interval $[a, b]$.

$$\text{Average Value} = \lim_{n \longrightarrow \infty} \frac{1}{b-a}\sum_{j=0}^{n-1} f(x_j)\triangle x = \frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 4 *If f is a continuous function on the closed interval $[a, b]$, then there is a number in $[a.b]$ where f is equal to its average value on this interval.*

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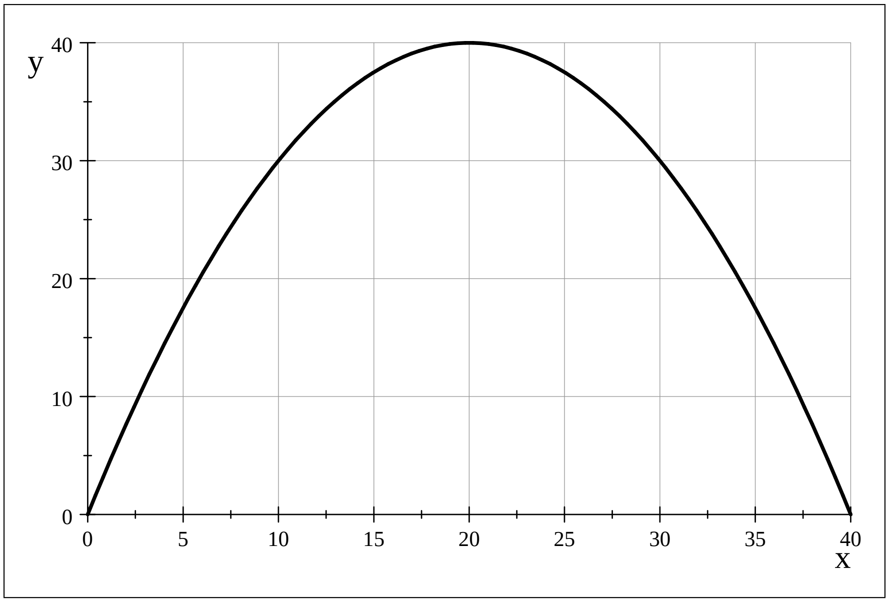
This result is often called the *Mean Value Theorem for Integrals*. It has a unique interpretation when the function f is positive as well as continuous on the interval $[a, b]$. In this case, the Mean Value Theorem for Integrals tells us

- There is a rectangle with base $b - a$ whose area is exactly equal to $\int_a^b f(x)dx$.

Example 5 *Suppose that a landowner wants to level a hillside to make a flat parking lot. A cross section showing the height of the hillside is shown below. To what approximate height should the owner lower the hill so that the debris exacty fills the space on either side?*



Solution. Since the landowner wants to create a flat space with no waste, the best way to accomplish this is to create a rectangle with the same base as the hill and the same area. According to the Mean Value Theorem for Integrals, the height of this rectangle is the average value for the height function h on this interval. The average value for h is

$$c = \frac{1}{40}\int_0^{40} h(x)dx$$

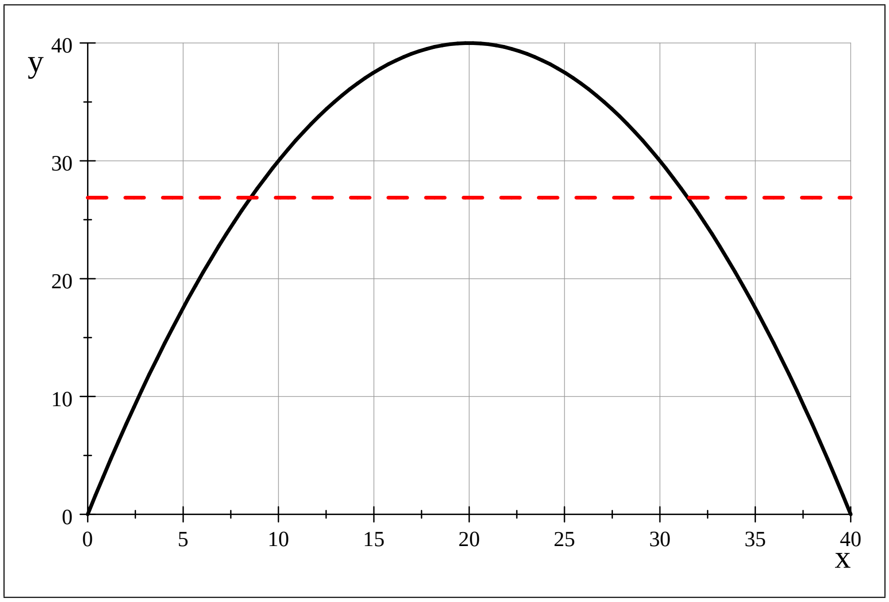
We will use a trapezoid estimate for this number. Dividing the interval $[0, 40]$ into sixteen subintervals gives us $\triangle x = 2.5$. This time, we will construct a table showing the endpoints of the subintervals and the approximate values of h at these numbers. (We have omitted $x_0 = 0$ and $x_{16} = 40$ to save space, since $h(0) = h(40) = 0$.)

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x_j	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5	25.0	27.5	30.0	32.5	35.0	37.5
$h(x_j)$	10	18	25	30	35	38	39	40	39	38	35	30	25	18	10

Only the bottom row of the table is needed for the trapezoid estimate. We know

$$\begin{aligned}\int_0^{40} h(x)dx &\approx \frac{2.5}{2}[0 + 20 + 36 + 50 + 60 + 70 + 76 + 78 + 80 + 78 + 76 + 70 + 60 + 50 + 36 + 20 + 0] \\ &= 1075\end{aligned}$$

Consequently, the average value is $c = 1075/40 = 26.875$. If the landowner lowers the hill to approximately 26.875 feet, the debris will just fill the void on either side of the hill, creating a flat surface. The dashed line in the diagram below shows the approximate level to which the hill should be lowered.



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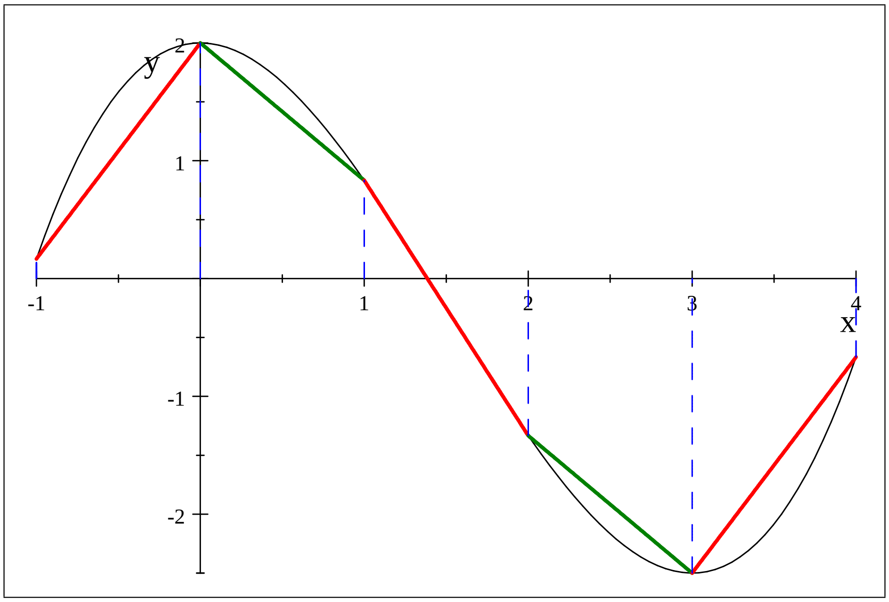
Problem 4. Use the trapezoid estimate with six subintervals to estimate the average value of $f(x) = x + \sin(\pi x)$ on the interval $[3, 5]$.

Let f be a function defined on a closed interval $[a, b]$. The *arc-length* of the graph of f on this interval is defined to be the actual length of the curve represented by the graph. (Imagine the graph as a piece of limp string. The arc-length is the number you would get if you took the graph, laid it flat on a table, and measured its length.)

Example 6 *Suppose that f is a differentiable function on a closed interval $[a, b]$. Show that the arc-length of the graph of f on this interval is given by*

$$\mathcal{L} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Solution. We will establish this formula by creating Riemann sums. To begin, suppose we have a function f that is differentiable on a closed interval $[a, b]$. We begin by dividing the interval into n subintervals of equal width $\triangle x$ as in the previous section. However, instead of building rectangles on these subintervals, we will construct the secant lines connecting their endpoints. Doing so gives us a curve composed of n straight line segments joined end to end (these are called *polygonal curves*). The diagram below shows an example of this.



The arc-length of the polygonal curve will approximate the arc-length of the function f, and this approximation will improve the larger we let n become (since this will cause $\triangle x$ to decrease, making the subintervals thinner). Since the polygonal curve is composed of secant lines to f, we can compute the length of each piece using the distance formula. Select a subinterval $I_j = [x_j, x_{j+1}]$ at random. The width of this subinterval is $\triangle x$. The endpoints of the secant line we constructed on this subinterval are $(x_j, f(x_j))$ and $(x_{j+1}, f(x_{j+1}))$. Consequently, the length of this secant line will be

$$\begin{aligned} d_j &= \sqrt{(x_{j+1}-x_j)^2 + (f(x_{j+1}) - f(x_j))^2} \\ &= \sqrt{(x_{j+1}-x_j)^2 \left[1 + \frac{(f(x_{j+1}) - f(x_j))^2}{(x_{j+1}-x_j)^2}\right]} \\ &= \sqrt{1 + \left[\frac{f(x_{j+1}) - f(x_j)}{x_{j+1}-x_j}\right]^2}\,(x_{j+1}-x_j) \end{aligned}$$

Now, the quotient under the radical is the slope of the secant line on the interval I_j; since f is differentiable on this interval, the Mean Value Theorem tells us that there is a number x_j^* in this interval such that

$$f'(x_j^*) = \frac{f(x_{j+1}) - f(x_j)}{x_{j+1}-x_j}$$

Therefore, since $\triangle x = x_{j+1} - x_j$ by definition (since it is the width of I_j), we see that the length of the secant line on this one subinterval is

$$d_j = \sqrt{1 + \left[f'(x_j^*)\right]^2}\,\triangle x$$

Consequently, the arc-length of the polygonal curve is the sum of these n lengths; and we know that the arc-length $\mathcal{L}$ of the function f is approximated by this sum. In particular, we know

$$\mathcal{L} \approx \sum_{j=0}^{n-1} \sqrt{1 + \left[f'(x_j^*)\right]^2}\,\triangle x \qquad \Longrightarrow \qquad \mathcal{L} = \lim_{n\longrightarrow\infty} \sum_{j=0}^{n-1} \sqrt{1 + \left[f'(x_j^*)\right]^2}\,\triangle x = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\,dx$$

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Example 7 *Use the formula from the previous example and a trapezoid estimate to approximate the arc-length of the function* $f(x) = \ln(x)$ *on the interval* $[1, 3]$.

Solution. According the the arc-length formula, we know that the arc-length of f on the interval $[1,3]$ is given by

$$\mathcal{L} = \int_1^3 \sqrt{1+[f'(x)]^2}dx = \int_1^3 \sqrt{1+\left[\frac{1}{x}\right]^2}dx = \int_1^3 \frac{\sqrt{x^2+1}}{x}dx$$

Unfortunately, we have no way to evaluate this definite integral exactly at this stage (this will change). As directed, we will approximate it using a trapezoid estimate. We are free to choose the number of estimating trapezoids we use, so let's use six. This means $n = 6$ in the trapezoid estimate, and $\triangle x = 1/3$. Our sequence of points for creating the subintervals is therefore

$$x_0 = 1 \qquad x_1 = \frac{4}{3} \qquad x_2 = \frac{5}{3} \qquad x_3 = 2 \qquad x_4 = \frac{7}{3} \qquad x_5 = \frac{8}{3} \qquad x_6 = 3$$

The trapezoid estimate for the arc-length is therefore

$$\begin{aligned}
\int_1^3 \frac{\sqrt{x^2+1}}{x}dx &\approx \frac{1}{2}\sum_{j=0}^{5}\left[\frac{\sqrt{(x_j)^2+1}}{x_j} + \frac{\sqrt{(x_{j+1})^2+1}}{x_{j+1}}\right]\left(\frac{1}{3}\right) \\
&= \frac{1}{6}\left(\frac{\sqrt{(1)^2+1}}{1} + 2\frac{\sqrt{(4/3)^2+1}}{4/3} + 2\frac{\sqrt{(5/3)^2+1}}{5/3} + 2\frac{\sqrt{(2)^2+1}}{2} + 2\frac{\sqrt{(7/3)^2+1}}{7/3}\right) \\
&\quad + \frac{1}{6}\left(2\frac{\sqrt{(8/3)^2+1}}{8/3} + \frac{\sqrt{(3)^2+1}}{3}\right) \\
&\approx 1.35975
\end{aligned}$$

****

Problem 5. Use a trapezoid estimate with six subintervals to approximate the arc-length of the curve defined by $f(x) = x^2$ on the interval $1 \le x \le 4$.

There is a subtle distinction between the definite integral defined above and the notion of net area we explored in Part 1. In the definition, we require that f be integrable on the interval *between* the numbers $x = a$ and $x = b$. There is no assumption in the definition that $a \le b$ like there is in the concept of net area. Now, if it is true that $a \le b$, then the definite integral is simply the net area between the graph of f and the x-axis on the interval $[a,b]$. However, if $b \le a$, then the number

$$\triangle x = \frac{b-a}{n}$$

is *negative*; and the definite integral provides the *negative* of the net area between the graph of f and the x-axis on the interval $[b,a]$.

- The definite integral of an integrable function f on the interval between the numbers $x = a$ and $x = b$ provides the *directed* net area between the graph of f and the x-axis on this interval, measured in the direction from $x = a$ to $x = b$. If this direction is to the right (which means $a \le b$), then the definite integral is the net area. If this direction is to the left (which means $b \leq a$) then the definite integral is the negative of the net area. We can summarize this succinctly with the equation

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

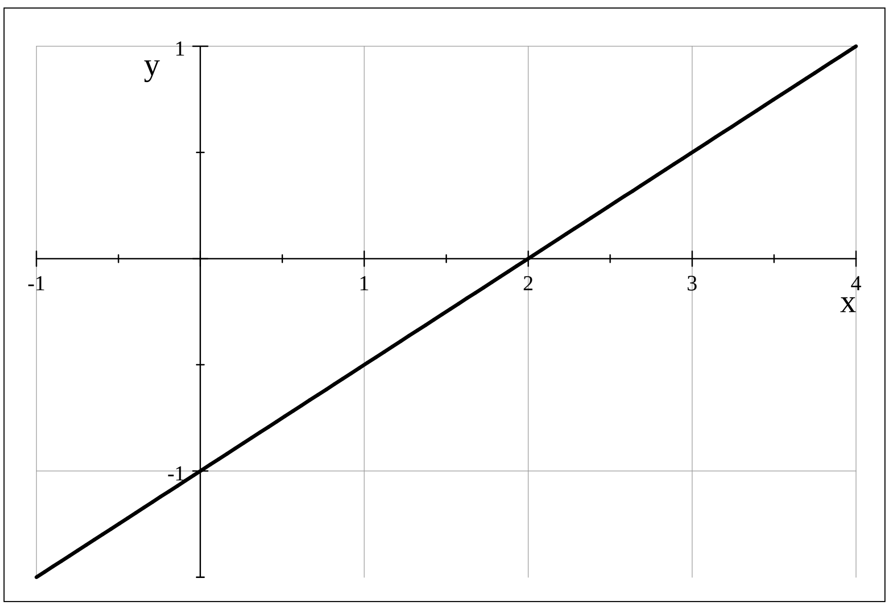
We will use this observation in a final application of the definite integral. In the next section, this application will show us how we can evaluate many definite integrals exactly.

Definition 8 *Let f be an integrable function on a closed interval $[a,b]$ and let c be a fixed number in this interval. For any number x in $[a,b]$, let*

$$F_c(t) = \int_c^t f(x)dx$$

denote the function which gives the directed net area for f on the interval between the numbers c and x, measured from c.

Example 9 *Let f be the function shown in the graph below. Use this graph to construct the functions F_{-1},and F_0, on the interval $[-1,3]$.*



Solution. We will construct the function F_{-1} first. By definition, the function gives the directed net area for f between the numbers -1 and x, measured from -1. First, note that $F_{-1}(-1) = 0$, since there must be zero net area on the (degenerate) interval $[-1,-1]$. What is the value of $F_{-1}(0)$? By definition, we know that $F_{-1}(0)$ is the directed net area for f on the interval $[-1,0]$, measured from -1. The region between the graph of f and the x-axis on this interval is a trapezoid; hence, we can compute the net area exactly:

$$F_{-1}(0) = \int_{-1}^0 f(x)dx = \frac{1}{2}\left[f(-1) + f(0)\right] = \frac{1}{2}\left[-\frac{3}{2} - 1\right](1) = -\frac{5}{4}$$

We can continue to evaluate F_{-1} in this way. Simply select numbers x in the interval $[-1,3]$ and compute the directed net area for f on the interval $[-1,x]$, measured from -1. Since we are measuring in the "positive" direction, we will simply be computing the net area. Observe that since thr graph of f is a straight line, all of the regions involved are triangles or trapezoids. Consequently, we can compute the net areas exactly. For example,

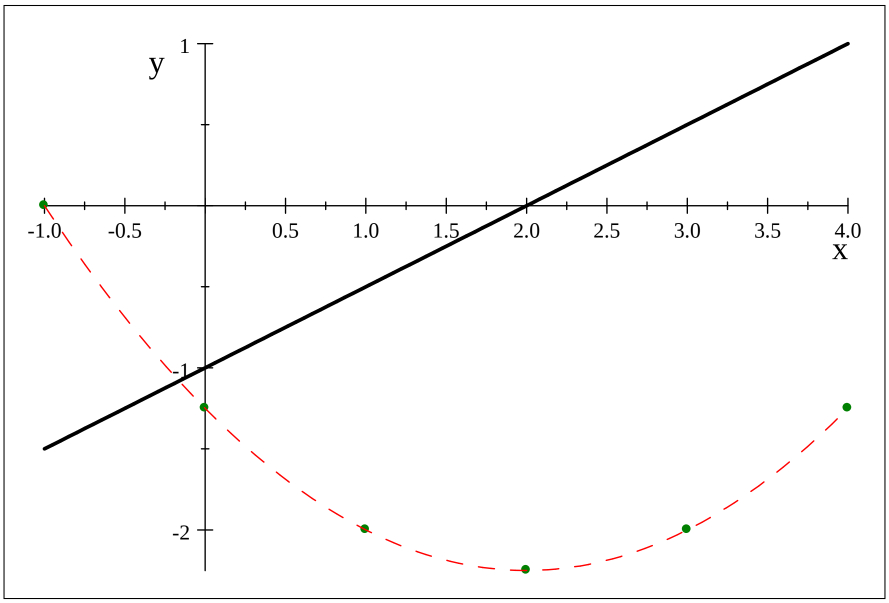
$$F_{-1}(1) = \int_{-1}^1 f(x)dx = \frac{1}{2}\left[f(-1) + f(1)\right](2) = \frac{1}{2}\left[-\frac{3}{2} - \frac{1}{2}\right](2) = -2$$

$$F_{-1}(2) = \int_{-1}^{2} f(x)dx = \frac{1}{2}\left[f(-1)\right](3) = \frac{1}{2}\left[-\frac{3}{2}\right](3) = -\frac{9}{4}$$

$$F_{-1}(3) = \int_{-1}^{3} f(x)dx = \int_{-1}^{2} f(x)dx + \int_{2}^{3} f(x)dx = -\frac{9}{4} + \frac{1}{2}\left[f(3)\right](1) = -\frac{9}{4} + \frac{1}{4} = -2$$

$$F_{-1}(4) = \int_{-1}^{4} f(x)dx = \int_{-1}^{3} f(x)dx + \int_{3}^{4} f(x)dx = -2 + \frac{1}{2}\left[f(3) + f(4)\right](1) = -2 + \frac{3}{4} = -\frac{5}{4}$$

These computations give us a few ordered pairs on the graph of F_{-1}, namely $(-1, 0)$, $(0, -5/4)$, $(1, -2)$, $(2, -9/4)$, $(3, -2)$, and $(4, -5/4)$. We can find as many additional ordered pairs as we like simply by choosing more values for x. However, these pairs give us a rough idea for what the graph of F_{-1} looks like:



To construct the graph for the function F_0, we use the same approach, only this time we are measuring net area from the number 0 instead of the number -1. This time, we know that $F_0(0) = 0$. What can we say about $F_0(-1)$? We are measuring the net area between the graph of f and the x-axis on the interval $[-1, 0]$ — only this time we are moving in the "negative" direction since $-1 < 0$ and we are measuring from 0. This means that $F_0(-1)$ is actually the negative of this net area. In symbols, we see that

$$F_0(-1) = \int_{0}^{-1} f(x)dx = -\int_{-1}^{0} f(x)dx = \frac{5}{4}$$

What about $F_0(1)$? In this case, we are measuring net area in the "positive" direction, since $0 < 1$ and we are measuring from 0. Thus, $F_0(1)$ is the true net area between the graph of f and the x-axis on the interval $[0, 1]$. In particular,

$$F_0(1) = \int_{0}^{1} f(x)dx = \frac{1}{2}\left[f(0) + f(1)\right](1) = -\frac{3}{4} \qquad \text{(The region is a trapezoid.)}$$

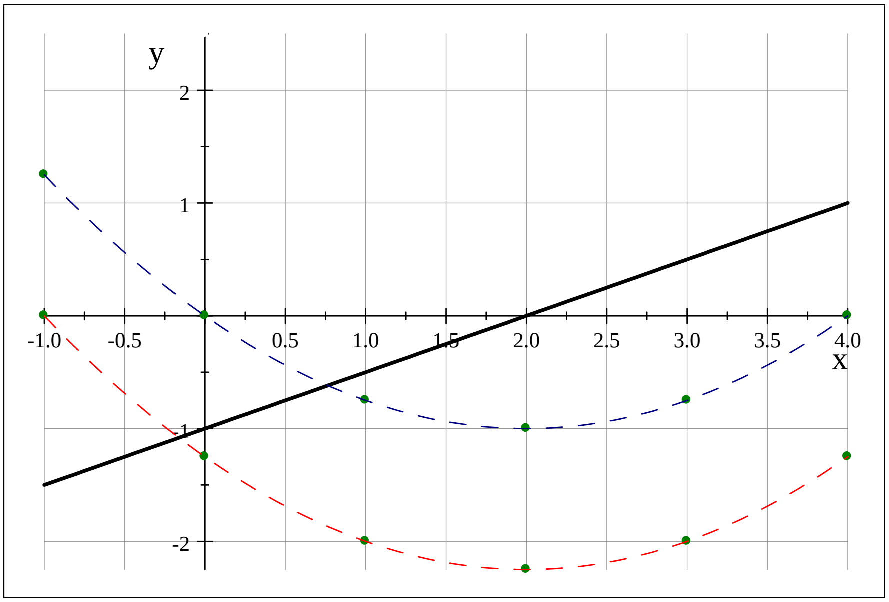
Proceeding in the same way, we find that

$$F_0(2) = \int_{0}^{2} f(x)dx = \frac{1}{2}\left[f(0)\right](2) = -1 \qquad \text{(The region is a right triangle.)}$$

$$F_0(3) = \int_{0}^{3} f(x)dx = \int_{0}^{2} f(x)dx + \int_{1}^{2} f(x)dx = -1 + \frac{1}{2}\left[f(3)\right](1) = -1 + \frac{1}{4} = -\frac{3}{4}$$

$$F_0(4) = \int_{0}^{4} f(x)dx = \int_{0}^{3} f(x)dx + \int_{3}^{4} f(x)dx = -\frac{3}{4} + \frac{1}{2}\left[f(3) + f(4)\right](1) = -\frac{-3}{4} + \frac{3}{4} = 0$$

These computations give us a few points on the graph of F_0, namely $(-1, 5/4)$, $(0, 0)$, $(1, -3/4)$, $(2, -1)$, $(3, -3/4)$, and $(4, 0)$. The diagram below shows a sketch of F_0 along with a sketch of F_{-1}.



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Notice how similar the the graphs F_0 and F_{-1} are in the last diagram. In fact, a quick check shows that one is simply a vertical translation of the other — indeed, $F_0 = F_{-1} + 5/4$.

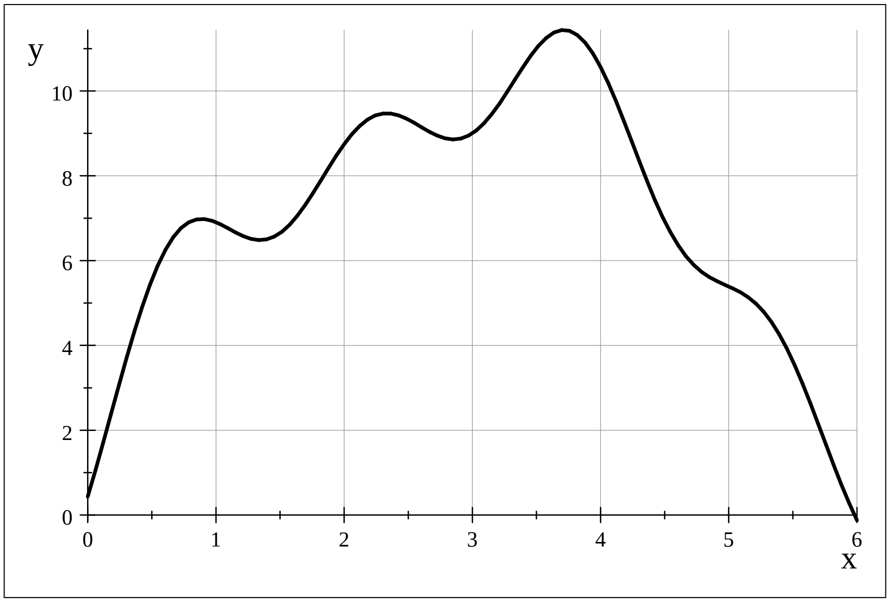
Problem 6. Following the previous example, sketch the graph of $F_1(t) = \int_1^t f(x)dx$ on the grid above.

- $F_1(-1) = \int_1^{-1} f(x)dx =$
- $F_1(0) = \int_1^{0} f(x)dx =$
- $F_1(1) = \int_1^{1} f(x)dx =$
- $F_1(2) = \int_1^{2} f(x)dx =$
- $F_1(3) = \int_1^{3} f(x)dx =$
- $F_1(4) = \int_1^{4} f(x)dx =$

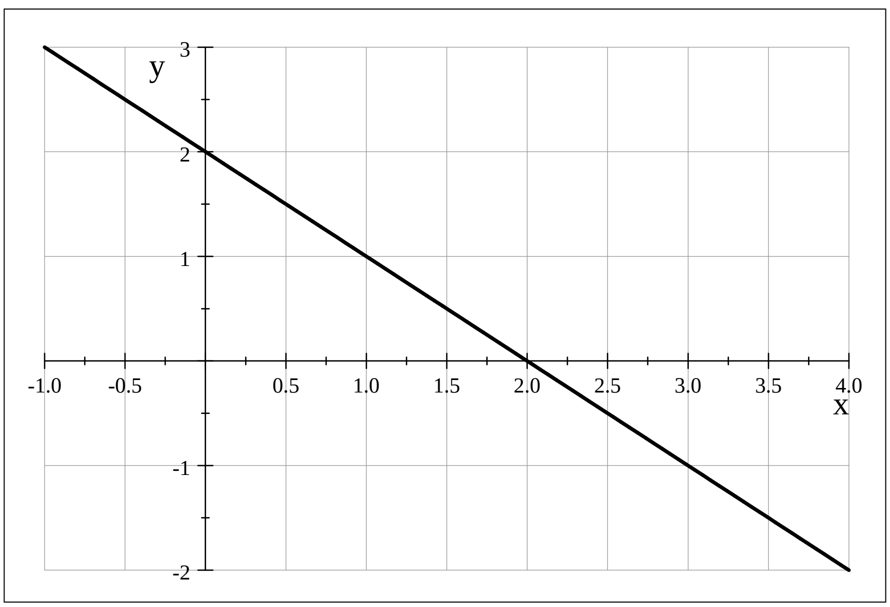
EXERCISES FOR PART 2

1. Use a right-hand estimate with six subintervals to approximate the average value of $f(x) = \sqrt{1+2x}$ on the interval $[1, 4]$.

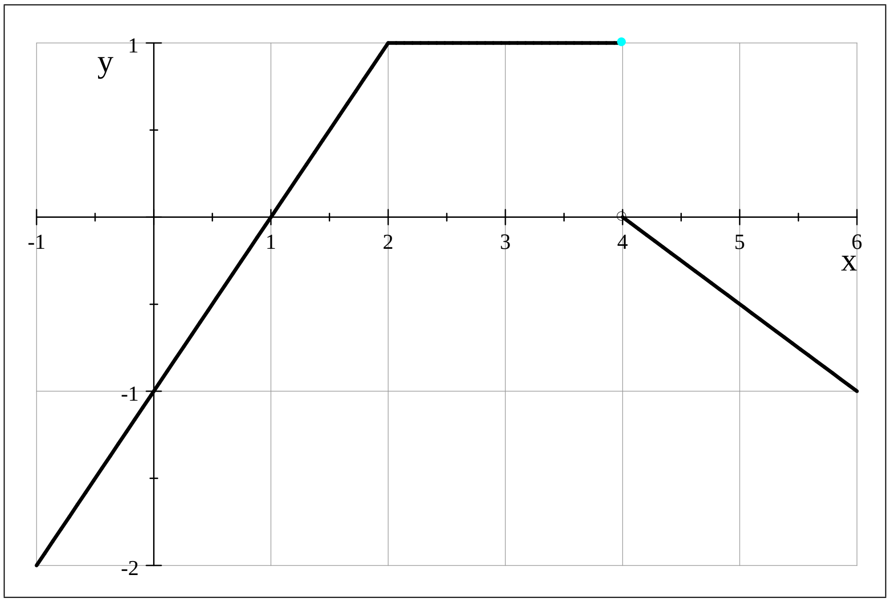
2. A geologist needs to estimate the average height for the entrance to a cave. The diagram below shows the height h of the entrance above the cave floor (both x and y are measured in feet). Use a trapezoid estimate with twelve subintervals to approximate the average height. Estimate the values of h to the nearest tenth.



3. The graph of a function f is shown below. Use it to construct the functions F_0 and F_3.



4. The graph of a function f is shown below. Use it to construct the functions F_1 and F_4.



5. Use a right-hand estimate with six subintervals to approximate the average value of $f(x) = \sqrt{1+2x}$ on the interval $[1, 4]$.

6. Use a left-hand estimate with six subintervals to approximate the arc-length of $f(x) = \cos(x)$ on the interval $[0, \pi/4]$.

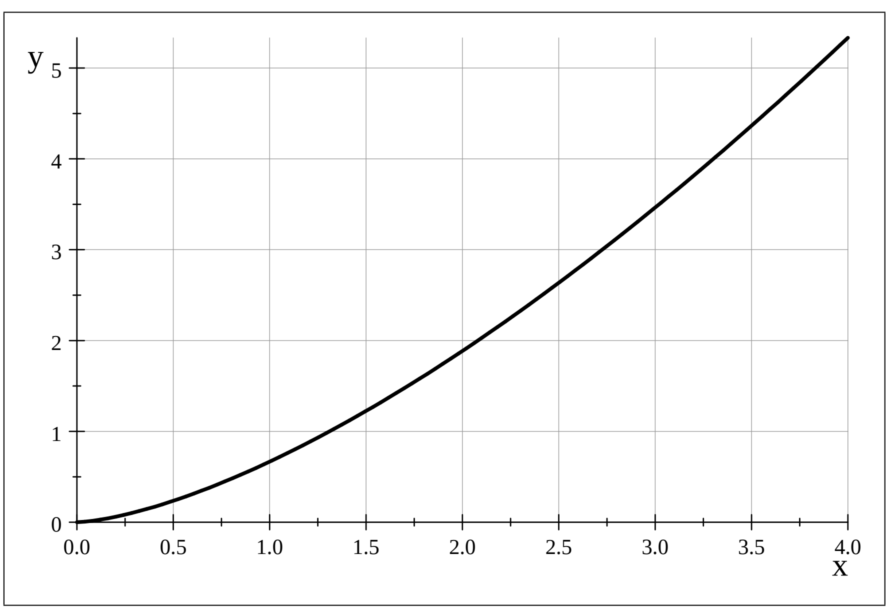
7. Use a trapezoid estimate with five subintervals to approximate the arc-length of $f(x) = x^3$ on the interval $[1, 3]$.

8. Let $f(x) = \sqrt{1-x^2}$ denote the arc of the circle $x^2 + y^2 = 1$ above the x-axis. Construct the definite integral which measures the arc-length of f on the interval $[0, 3/4]$ and then use a right-hand estimate with six subintervals to approximate this length.

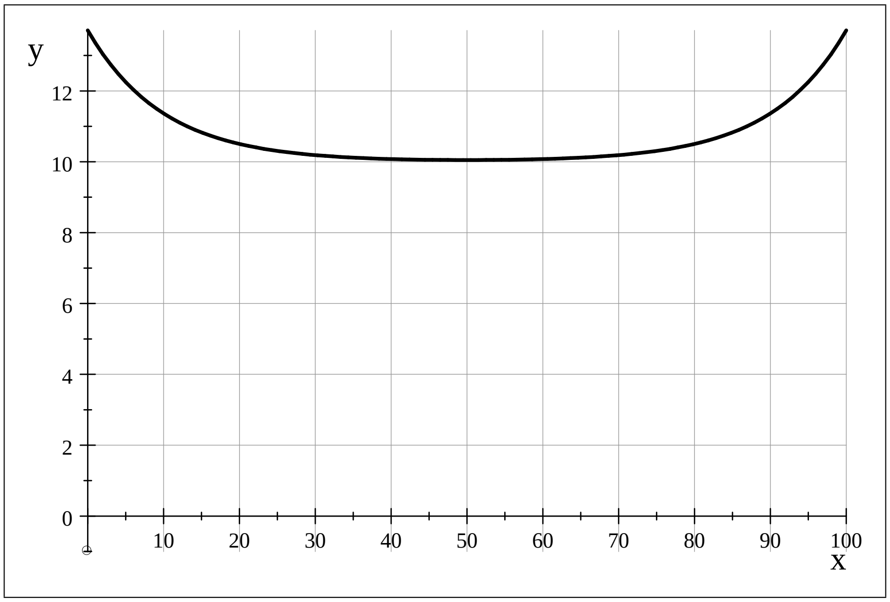
9. A book is left open on a table in a humid room, and one page curls up into the shape shown below. Both x and y are measured in inches.

   (a) Estimate the graph of the derivative for the curve.

   (b) Use your approximate derivative and a trapezoid estimate with four subintervals to approximate the length of the page.

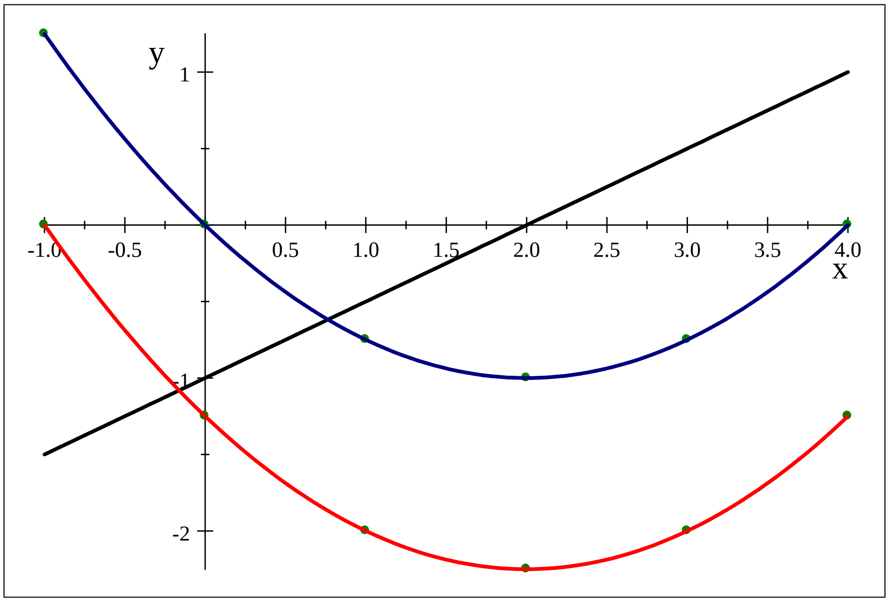


10. A cable hanging between two telephone poles sags into the shape shown below (x and y are both measured in feet). Use a trapezoid estimate with five subintervals to approximate the average height of the cable above the ground.



3 Part 3 — The Fundamental Theorems of Calculus

We will begin this section by taking a closer look at some of the integral-based functions we created in the last section. In particular, let's take another look at the functions F_0 and F_{-1} we constructed in in the previous section.



We have already noticed that these two curves are vertical translations of each other. A closer look suggests something more. When the graphs of both F_0 and F_{-1} are falling, the function f from which they are constructed is *below* the x-axis. When the graphs of both F_0 and F_{-1} are rising, the function f from which they are constructed is *above* the x-axis. Furthermore, the graphs of both F_0 and F_{-1} have a local minimum at $x = 2$; and this is precisely where the graph of f crosses the x-axis. In other words, it looks like the function f is actually the *derivative* of both F_0 and F_{-1}.

If you check the other integral-based functions you constructed, you will see the same apparent relationship — the function f used to create the integral-based functions appears to be their derivative. This observation suggests something profound:

- Up to a vertical translation, we can recreate a function F from its derivative f using the integral-based functions discussed in the previous section.

This observation is known as the *First Fundamental Theorem of Calculus*. We will take a moment to prove a special case of it, since its proof uses several of our earlier results.

Theorem 10 *If f is a continuous function on an interval $[a, b]$ and if c is any number in this interval, then*

$$F_c(x) = \int_c^x f(t)dt$$

is an antiderivative for the function f.

Proof. First, note that, by definition, we know

$$\begin{aligned}
\frac{d}{dx}\left[F_c(x)\right] &= \lim_{\triangle x \longrightarrow 0} \frac{F_c(x+\triangle x) - F_c(x)}{\triangle x} \\
&= \lim_{\triangle x \longrightarrow 0} \left(\frac{1}{\triangle x}\right)\left[\int_c^{x+\triangle x} f(t)dt - \int_c^x f(t)dt\right] \\
&= \lim_{\triangle x \longrightarrow 0} \left(\frac{1}{\triangle x}\right)\left[\int_c^{x+\triangle x} f(t)dt + \int_x^c f(t)dt\right] \\
&= \lim_{\triangle x \longrightarrow 0} \left(\frac{1}{\triangle x}\right)\left[\int_x^{x+\triangle x} f(t)dt\right]
\end{aligned}$$

Now, the expression

$$\frac{1}{\triangle x}\int_x^{x+\triangle x} f(t)dt$$

is nothing more than the average value of the function F_c on the closed interval between the numbers x and $x + \triangle x$. Therefore, since f is continuous on the interval $[a, b]$ and thus also continuous on the subinterval between the numbers x and $x + \triangle x$, we know by the Mean Value Theorem for Integrals that there exists a value b_x in this interval such that

$$f(b_x) = \frac{1}{\triangle x}\int_x^{x+\triangle x} f(t)dt$$

Therefore, we see that

$$\frac{d}{dx}\left[F_c(x)\right] = \lim_{\triangle x \longrightarrow 0} \left(\frac{1}{\triangle x}\right)\left[\int_x^{x+\triangle x} f(t)dt\right] = \lim_{\triangle x \longrightarrow 0} f(b_x) = f(x)$$

The last equality comes from the fact that b_x lies in the interval determined by between the numbers x and $x + \triangle x$; so, as $\triangle x$ approaches 0, the number b_x must approach x.

****

The Fundamental Theorem of Calculus actually is true even when f is not continuous on the interval $[a, b]$ (as long as it is integrable on this interval). The proof of this fact, however, is beyond the scope of this course.

Example 11 *Find the derivative of the function F defined by*

$$F(x) = \int_2^x t^2\sqrt{1 - \sin(t)}dt$$

Solution. This problem is much easier that it might appear, thanks to the First Fundamental Theorem of Calculus. This theorem tells us that F is an antiderivative for the function $f(t) = t^2\sqrt{1 - \sin(t)}$. Therefore,

$$F'(x) = \frac{d}{dx}\left[\int_2^x t^2\sqrt{1 - \sin(t)}dt\right] = x^2\sqrt{1 - \sin(x)}$$

****

Problem 6. Use the First Fundamental Theorem to find $F'(x)$ when F is defined by the following formulas.

$$\text{(a)}\;\; F(x) = \int_3^x \frac{1-e^r}{r}dr \qquad \text{(b)}\;\; F(x) = \int_3^x t^3 dt \qquad \text{(b)}\;\; F(x) = \int_3^x w\tan(\pi w)dw$$

The next result, often called the *Second Fundamental Theorem of Calculus*, is one of the most useful results in mathematics.

Theorem 12 *Let f be a continuous function on a closed interval $[a,b]$. If F is any antiderivative for f on $[a,b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. Suppose F is any antiderivative for f on $[a,b]$. Divide this interval into n equally spaced subintervals of width $\triangle x$, and let

$$x_0 = a \qquad x_1 = a + \triangle x \qquad \ldots \qquad x_{n-1} = a + (n-1)\triangle x \qquad x_n = b$$

be the endpoints of these subintervals. Observe that

$$F(b) - F(a) = F(b) + F(x_{n-1}) - F(x_{n-1}) + \ldots + F(x_1) - F(x_1) - F(a) = \sum_{j=0}^{n-1}[F(x_{j+1}) - F(x_j)]$$

Now, since F is an antiderivative for f on $[a,b]$, we know that $F'(x) = f(x)$ for all numbers x in $[a,b]$. In particular, this tells us that F is differentiable on the open interval (a,b). On each subinterval $[x_j, x_{j+1}]$, the difference quotient

$$\frac{F(x_{j+1}) - F(x_j)}{\triangle x}$$

represents the average rate of change for F. Now, since F is differentiable, when $\triangle x$ is small, we know the average rate of change for F on the interval $[x_j, x_{j+1}]$ is approximately equal to $F'(x_j)$; and this approximation gets better and better the smaller $\triangle x$ becomes. In symbols, we know

$$\frac{F(x_{j+1}) - F(x_j)}{\triangle x} \approx F'(x_j) = f(x_j)$$

Since $F(b) - F(a)$ does not depend on the number of subintervals we choose, we know that

$$\begin{aligned}
F(b) - F(a) &= \lim_{n\longrightarrow\infty}[F(b) - F(a)] \\
&= \lim_{n\longrightarrow\infty}\sum_{j=0}^{n-1}[F(x_{j+1}) - F(x_j)] \\
&= \lim_{n\longrightarrow\infty}\sum_{j=0}^{n-1}\left[\frac{F(x_{j+1}) - F(x_j)}{\triangle x}\right]\triangle x = \lim_{n\longrightarrow\infty}\sum_{j=0}^{n-1} f(x_j^*)\triangle x = \int_a^b f(x)dx
\end{aligned}$$

****

The Second Fundamental Theorem of Calculus tells us how to find the exact value of the definite integral for any function whose antiderivative family is known. This gives the Second Fundamental Theorem of Calculus great practical significance.

Example 13 *Use the Second Fundamental Theorem of Calculus to find the exact value for* $\int_{-1}^{2}\left(x^2-2x+1\right)dx$.

Solution. We know that the antiderivative family for $f(x)=x^2-2x+1$ is the set

$$\int\left(x^2-2x+1\right)dx=\frac{x^3}{3}-x^2+x+C$$

The Second Fundamental Theorem tells us that we may evaluate the definite integral directly by using *any one* of these antiderivatives. It does not matter which one we pick. For example, we could let

$$F(x)=\frac{x^3}{3}-x^2+x+8$$

According to the Second Fundamental Theorem,

$$\begin{aligned}\int_{-1}^{2}\left(x^2-2x+1\right)dx &= F(2)-F(-1)\\ &= \left[\frac{2^3}{3}-2^2+2+8\right]-\left[\frac{(-1)^3}{3}-(-1)^2+(-1)+8\right]=5\end{aligned}$$

****

The antidifferentiation constant C simply cancels out of the difference $F(b)-F(a)$; that is why it does not matter which antiderivative we pick to evaluate a definite integral. Consequently, it is customary (but not necessary) to let $C=0$ in this use of antiderivatives. Rather than identifying an antiderivative as a separate step and then using it to evaluate a definite integral like we did in the last example, it is also common to use the following notation

$$\begin{aligned}\int_{-1}^{2}\left(x^2-2x+1\right)dx &= \left.\frac{x^3}{3}-x^2+x+8\right|_{x=-1}^{x=-2} \qquad \text{Read “Evaluate from } -1 \text{ to } 2.\text{”}\\ &= \left[\frac{2^3}{3}-2^2+2+8\right]-\left[\frac{(-1)^3}{3}-(-1)^2+(-1)+8\right]=5\end{aligned}$$

Example 14 *Find the exact value of* $\int_{0}^{1/2}\sec^2(x)dx$.

Solution. We know that

$$\int\sec^2(x)dx=\tan(x)+C$$

Therefore, we also know that

$$\begin{aligned}\int_{0}^{1/2}\sec^2(x)dx &= \tan(x)|_{x=0}^{x=1/2}\\ &= \tan\left(\frac{1}{2}\right)-\tan(0)=\tan\left(\frac{1}{2}\right)\end{aligned}$$

****

Problem 7. Use the Second Fundamental Theorem to determine the exact value of $\int_1^4 \left(3t^2 - 2t + 1\right) dt$.

Problem 8. Use the Second Fundamental Theorem to determine the exact value of $\int_6^1 \frac{1}{x} dx$.

Example 15 *Find the exact value of* $\int_{x=2}^{x=4} \frac{x-1}{x^2 - 2x + 1} dx$.

Solution. Finding an antiderivative in this case will require a substitution. Observe that

$$u = x^2 - 2x + 1 \qquad \Longrightarrow \qquad \frac{du}{dx} = 2x - 2 \qquad \Longrightarrow \qquad \frac{1}{2}\frac{du}{dx} = x - 1$$

$$\begin{aligned} \int \frac{x-1}{x^2 - 2x + 1} dx &= \int \frac{1}{x^2 - 2x + 1} [x-1] dx \\ &= \int \frac{1}{u} \left[\frac{1}{2} \frac{du}{dx} \right] dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \ln \sqrt{|x^2 - 2x + 1|} + C \end{aligned}$$

Therefore, we know that

$$\begin{aligned} \int_{x=2}^{x=4} \frac{x-1}{x^2 - 2x + 1} dx &= \ln \sqrt{|x^2 - 2x + 1|} \Big|_{x=2}^{x=4} \\ &= \ln \sqrt{4^2 - 2(4) + 1} - \ln \sqrt{2^2 - 2(2) + 1} = \ln \sqrt{9} - \ln \sqrt{1} = \ln 3 \end{aligned}$$

****

In the previous examples, we evaluated the definite integral by first constructing the antiderivative family for the function, then selecting one of the antiderivatives (by choosing a specific value for the integration constant), then applying the Second Fundamental Theorem of Calculus. If we are careful, it is possible to combine all of these into one process.

Example 16 *Find the exact value of* $\displaystyle\int_{x=1}^{x=5} x\sqrt{1+2x^2}dx.$

Solution. We can work this problem the same way we worked the last example, but this time we will combine the antidifferentation and the application of the Second Fundamental Theorem into one step. Pay careful attention to the process.

$$\begin{aligned}
\int_{x=1}^{x=5} x\sqrt{1+2x^2}dx &= \int_{x=1}^{x=5} \sqrt{1+2x^2}\,[x]\,dx && \text{Let } u=1+x^2 \text{ so } \frac{1}{2}\frac{du}{dx}=x \\
&= \int_{u=1+2(1)^2}^{u=1+2(5)^2} \sqrt{u}\left[\frac{1}{2}\frac{du}{dx}\right]dx && \text{Replace every occurrance of } x \text{ with } u \\
&= \frac{1}{2}\int_{u=3}^{u=51}\sqrt{u}du \\
&= \frac{1}{3}u^{3/2}\Big|_{u=3}^{u=51} = \frac{1}{3}\left(51\sqrt{51}-3\sqrt{3}\right)
\end{aligned}$$

Notice that we did not switch back to the variable x once we found an antiderivative. There is no need since we have converted every occurrance of x into some function of u. When combining the two steps of the evaluation process, it is imperative that all occurrances of the original variable be changed over to the substitution variable (when a substitution is required).

****

Example 17 *Find the exact value of* $\displaystyle\int_{x=0}^{x=\pi/2}\left[x\cos(x^2)-x\right]dx.$

Solution. In this case we have a difference of two functions. We will need a substitution to find an antiderivative for one, but we do not need a substitution for the other. We therefore break the definite integral up and handle each piece seperately.

$$\begin{aligned}
\int_{x=0}^{x=\sqrt{\pi/2}}\left[x\cos(x^2)-x\right]dx &= \int_{x=0}^{x=\sqrt{\pi/2}} x\cos(x^2)dx - \int_{x=0}^{x=\sqrt{\pi/2}} xdx \\
&= \int_{x=0}^{x=\sqrt{\pi/2}} \cos(x^2)\,[x]\,dx - \frac{x^2}{2}\Big|_{x=0}^{x=\sqrt{\pi/2}} && \text{Let } u=x^2 \text{ so } \frac{1}{2}\frac{du}{dx}=x \\
&= \int_{u=0}^{u=\pi/2}\cos(u)\left[\frac{1}{2}\frac{du}{dx}\right]dx - \frac{x^2}{2}\Big|_{x=0}^{\sqrt{\pi/2}} && \text{Replace all } x \text{ with } u \text{ in first integral} \\
&= \frac{1}{2}\sin(u)\Big|_{u=0}^{u=\pi/2} - \frac{x^2}{2}\Big|_{x=0}^{x=\sqrt{\pi/2}} \\
&= \left[\frac{1}{2}\sin\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin(0)\right] - \left[\frac{(\sqrt{\pi/2})^2}{2}-\frac{0^2}{2}\right] = \frac{1}{2}-\frac{\pi}{4}
\end{aligned}$$

****

Problem 9. Find the exact value of $\displaystyle\int_1^3 \frac{1}{1+2t}\,dt$.

Problem 10. Find the exact value of $\displaystyle\int_4^0 r\left(r^2-3\right)^3 dr$.

Exercises for Part 3

Use the Second Fundamental Theorem of Calculus to find the exact value of the following definite integrals.

(1) $\displaystyle\int_0^1 x^2 dx$ (2) $\displaystyle\int_{\pi/2}^{3\pi/2} \cos(x)dx$ (3) $\displaystyle\int_1^3 (3x^2+2x+1)dx$

(4) $\displaystyle\int_0^5 (7\sqrt{t}+6)dt$ (5) $\displaystyle\int_{-2}^{-1} \left(u^2-1\right) du$ (6) $\displaystyle\int_1^3 (z^{-2/3}+4z)dz$

(7) $\displaystyle\int_0^\pi \sin(3x)dx$ (8) $\displaystyle\int_{\pi/2}^{3\pi/4} \frac{\cos(x)}{\sin(x)}dx$ (9) $\displaystyle\int_1^3 \frac{4x-4}{(2x^2-4x+6)^2}dx$

(10) $\displaystyle\int_2^5 \frac{2x-1}{x^2-x+4}dx$ (11) $\displaystyle\int_4^1 \frac{2x}{\sqrt{1+x^2}}dx$ (12) $\displaystyle\int_1^0 \frac{x^3}{1+x^4}dx$

(13) $\displaystyle\int_0^4 \left(\sin(\pi x)-x^2\right) dx$ (14) $\int_{-1}^3 \left(x\sqrt{1+x^2}+x\right) dx$ (15) $\displaystyle\int_1^6 \left(x^{-1/2}+\frac{x^2+2}{x^3+6x-1}\right) dx$

(16) Find the exact average value of $f(x)=\sqrt{3x-5}$ on the interval $[2,10]$.

(17) Find the exact average value of $f(x)=2x+x^{-2}$ on the interval $[1,4]$.

(18) If $F(x)=\displaystyle\int_1^x e^{-t}\sin(\pi t)dt$, then what is $F'(1/2)$?

(19) If $G(t)=\displaystyle\int_0^t \frac{u^2-2u+1}{u^3+1}du$, then what is $G'(3)$?

4 Answers to Exercises

Section 1

1. Left-Hand Estimate $(n = 4)$ $\displaystyle\int_1^3 v(t)dt \approx -188$ feet
2. Trapezoid Estimate $(n = 8)$ $\displaystyle\int_1^3 v(t)dt \approx -156$ feet
3. No; Doris starts out travelling *toward* the lake.
4. After approximately six minutes; At this time, Doris turns and begins travelling *away* from the lake.
5. Closest distance from lake: $\displaystyle 600 + \int_1^6 v(t)dt \approx 91$ feet
6. Left-Hand Estimate $(n = 6)$ $\displaystyle\int_0^{3/2} \cos(\pi x)dx \approx -.18$
7. Trapezoid Estimate $(n = 8)$ $\displaystyle\int_1^3 \ln(1 + x^2)dx \approx 3.14$
8. Right-Hand Estimate $(n = 5)$ $\displaystyle\int_{-1}^1 (3x - 1)^3 dx \approx -7.0$
9. Left-Hand Estimate $(n = 7)$ $\displaystyle\int_{-2}^0 \frac{1}{1 + x^2} dx \approx .992$
10. $\displaystyle\int_{-1}^1 f(x)dx = 3 + \frac{\pi}{4}$
11. $\displaystyle\int_0^{4.25} f(x)dx = \frac{41}{8} + \frac{\pi}{2}$
12. $\displaystyle\int_1^{3.5} f(x)dx = \frac{7}{2} + \frac{\pi}{4}$

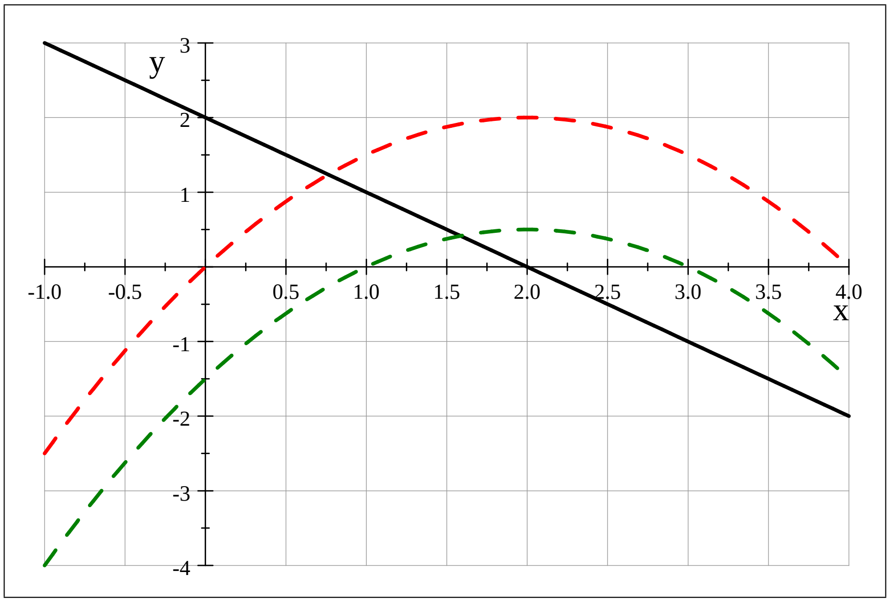
Section 2

1. Right-Hand Estimate $(n = 6)$ $\displaystyle\frac{1}{3}\int_1^4 \sqrt{1 + 2x}dx \approx 2.53$
2. Based on the graph, we obtain the following table of (approximate) heights

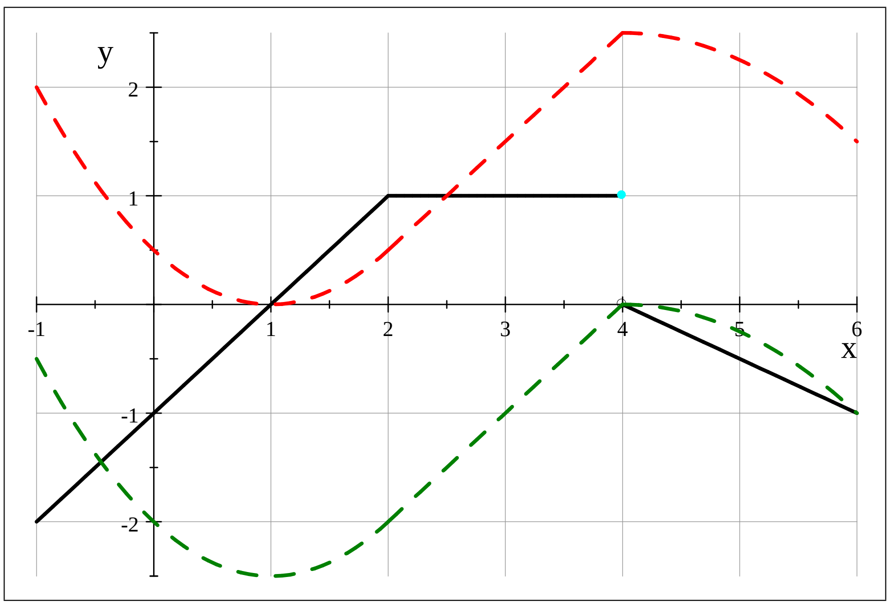
x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
$h(x)$	0.5	5.0	6.9	6.9	8.8	9.5	9.0	10.8	10.5	7.0	5.5	3.6	0.0

   Using Trapezoid Rule based on the table above, average height is $\displaystyle\frac{1}{6}\int_0^6 h(x)dx \approx 7$ feet

3. Here are approximate graphs for F_0 and F_3.



4. Here are the approximate graphs for F_1 and F_4.



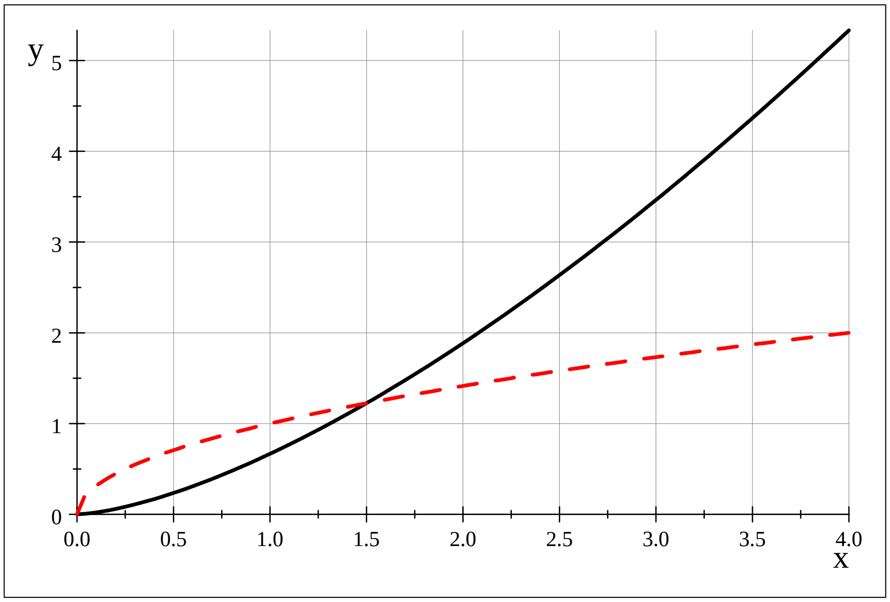
5. Right-Hand Estimate $(n = 6)$ $\frac{1}{3}\int_1^4 \sqrt{1+2x}dx \approx 2.53$

6. Left-Hand Estimate $(n = 6)$ $\int_0^{\pi/4} \sqrt{1+\sin^2(x)}dx \approx .84$

7. Trapezoid Estimate $(n = 5)$ $\int_1^3 \sqrt{1+9x^4}dx \approx 26.27$

8. Right-Hand Estimate $(n = 6)$ $\int_0^{3/4} \frac{1}{\sqrt{1-x^2}}dx \approx .88$

9. Dashed line shows estimated graph of derivative



Based on this graph, we obtain the following table of (approximate) values for f'

x	0.0	1.0	2.0	3.0	4.0
$f'(x)$	0.0	1.0	1.4	1.7	2.0

Using Trapezoid Rule based on the table above, the arc-length is

$$\int_0^4 \sqrt{1+[f'(x)]^2}dx \approx \left(\frac{1}{2}\right)\left[\sqrt{1}+2\sqrt{2}+2\sqrt{1+(1.4)^2}+2\sqrt{1+(1.7)^2}+\sqrt{5}\right] \approx 7.25 \text{ inches}$$

10. Based on the graph, we obtain the following table of (approximate) heights for the cable

x	0	20	40	60	80	100
$h(x)$	14.0	10.5	10.1	10.1	10.5	14.0

Using Trapezoid Rule based on the table above, the average height is $\dfrac{1}{100}\displaystyle\int_0^{100} h(x)dx \approx 11$ feet

Section 3

1. $\displaystyle\int_0^1 x^2 dx = \frac{1}{3}$

2. $\displaystyle\int_{\pi/2}^{3\pi/2} \cos(x)dx = -2$

3. $\displaystyle\int_1^3 (3x^2+2x+1)dx = \left(x^3+x^2+x\right)\Big|_{x=1}^{x=3} = 36$

4. $\displaystyle\int_0^5 (7\sqrt{t}+6)dt = \left(\frac{14}{3}t^{3/2}+6t\right)\Bigg|_{t=0}^{t=5} = 30+\frac{70\sqrt{5}}{3}$

5. $\displaystyle\int_{-2}^{-1} \left(u^2-1\right) du = \left(\frac{u^3}{3}-u\right)\Bigg|_{u=-2}^{u=-1} = \frac{4}{3}$

6. $\displaystyle\int_1^3 (z^{-2/3}+4z)dz = \left(3\sqrt[3]{z}+2z^2\right)\Big|_{z=1}^{z=3} = 3(\sqrt[3]{3}+5)$

7. $\displaystyle\int_0^{\pi} \sin(3x)dx = -\frac{1}{3}\cos(u)\Big|_{u=0}^{u=3\pi} = \frac{2}{3}$

8. $\displaystyle\int_{\pi/2}^{3\pi/4} \frac{\cos(x)}{\sin(x)}dx = \ln|u|\big|_{u=1}^{u=\sqrt{2}/2} = \frac{1}{2}\ln(2)$

9. $\displaystyle\int_1^3 \frac{4x-4}{(2x^2-4x+6)^2}dx = -\frac{1}{u}\Big|_{u=4}^{u=12} = \frac{1}{6}$

10. $\displaystyle\int_2^5 \frac{2x-1}{x^2-x+4}dx = \ln|u|\big|_{u=6}^{u=24} = \ln(4)$

11. $\displaystyle\int_4^1 \frac{2x}{\sqrt{1+x^2}}dx = 2\sqrt{u}\big|_{u=-17}^{u=2} = 2\sqrt{2}-2\sqrt{17}$

12. $\displaystyle\int_1^0 \frac{x^3}{1+x^4}dx = \frac{1}{4}\ln|u|\Big|_{u=2}^{u=1} = -\tfrac{1}{4}\ln(2)$

13. $\displaystyle\int_0^4 \left(\sin(\pi x)-x^2\right)dx = -\frac{1}{\pi}\cos(u)\Big|_{u=0}^{u=4\pi} - \frac{x^3}{3}\Big|_{x=0}^{x=4} = -\frac{64}{3}$

14. $\int_{-1}^3 \left(x\sqrt{1+x^2}+x\right)dx = \displaystyle\frac{u\sqrt{u}}{3}\Big|_{u=2}^{u=10} + \frac{x^2}{2}\Big|_{x=-1}^{x=3} = \frac{2}{3}\left(5\sqrt{10}-\sqrt{2}\right)+4$

15. $\displaystyle\int_1^6 \left(x^{-1/2}+\frac{x^2+2}{x^3+6x-1}\right)dx = 2\sqrt{x}\big|_{x=1}^{x=6} + \frac{1}{3}\ln|u|\Big|_{u=6}^{u=251} = 2\left(\sqrt{6}-1\right)+\frac{\ln(251)-\ln(6)}{3}$

16. Average Value is $\displaystyle\frac{1}{8}\int_2^{10}\sqrt{3x-5}dx = \frac{u\sqrt{u}}{36}\Big|_{u=1}^{u=25} = \frac{31}{9}$

17. Average Value is $\displaystyle\frac{1}{3}\int_1^4 \left(2x+x^{-2}\right)dx = \frac{1}{3}\left(x^2-\frac{1}{x}\right)\Big|_{u=1}^{u=25} = \frac{21}{4}$

18. If $\displaystyle F(x)=\int_1^x e^{-t}\sin(\pi t)dt$, then $\displaystyle F'(1/2)=\frac{1}{\sqrt{e}}$.

19. If $\displaystyle G(t)=\int_0^t \frac{u^2-2u+1}{u^3+1}du$, then $\displaystyle G'(3)=\frac{1}{7}$.