PRACTICE FOR THE NET CHANGE THEOREM

Problem 1: Find the exact value for $\int_1^4 \left(t^2 - \frac{3}{t^2}\right) dt$.

There is no viable u-substitution for the function appearing in the definite integral. Observe

$$\begin{aligned}
\int_1^4 \left(t^2 - \frac{3}{t^2}\right) dt &= \int_1^4 t^2 dt - 3\int_1^4 t^{-2} dt \\
&= \left.\frac{t^3}{3}\right|_{t=1}^{t=4} + \left.\left(\frac{3}{t}\right)\right|_{t=1}^{t=4} \\
&= \left(\frac{64}{3} - \frac{1}{3}\right) + 3\left(\frac{1}{4} - 1\right) \\
&= 21 - \frac{9}{4}
\end{aligned}$$

Problem 2: Find the exact value for $\int_0^2 \frac{z-1}{(z^2-2z+1)^3} dz$.

In this problem, let $u = z^2 - 2z + 1$. We then have $\frac{du}{dz} = 2(z-1)$ which tells us that $\left(\frac{1}{2}\right)\frac{du}{dz} = z - 1$. Now,

$$\begin{aligned}
\int_0^2 \frac{z-1}{(z^2-2z+1)^3} dz &= \int_{z=0}^{z=2} \frac{1}{(z^2-2z+1)^3} [z-1]\, dz \\
&= \int_{u=1}^{u=1} \frac{1}{u^3}\left[\frac{1}{2}\frac{du}{dz}\right] dz \\
&= \frac{1}{2}\int_{u=1}^{u=1} u^{-3} du \\
&= \left. -\frac{u^{-2}}{4}\right|_{u=1}^{u=1} \\
&= 0
\end{aligned}$$

Problem 3: Find the average value of the function $f(x) = x - 3\sin(x)$ on the interval $\frac{\pi}{2} \leq x \leq \pi$.

The average value of a function f on the interval $a \leq x \leq b$ is given by the formula $A = \frac{1}{b-a}\int_a^b f(x)dx$. With this in mind,

$$\begin{aligned}
A &= \frac{1}{\pi - \pi/2}\int_{\pi/2}^{\pi} (x - 3\sin(x))\, dx \\
&= \frac{2}{\pi}\left(\int_{\pi/2}^{\pi} x dx - 3\int_{\pi/2}^{\pi} \sin(x) dx\right) \\
&= \frac{2}{\pi}\left[\left.\frac{x^2}{2}\right|_{x=\pi/2}^{x=\pi} + 3\cos(x)|_{x=\pi/2}^{x=\pi}\right] \\
&= \frac{2}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right) + 3\left(\cos(\pi) - \cos\left(\frac{\pi}{2}\right)\right)\right] \\
&= \frac{2}{\pi}\left[\frac{3\pi^2}{8} - 3\right]
\end{aligned}$$

Problem 4: Find the net area of the function $f(p) = p\left(3p^2 - 4\right)^{1/3}$ on the interval $0 \leq p \leq 2$.

The net area is given by the formula $\int_0^2 p\left(3p^2-4\right)^{1/3} dp$. Let $u = 3p^2 - 4$ so that $\frac{du}{dp} = 6p$.

$$\begin{aligned}
\int_0^2 p\left(3p^2-4\right)^{1/3} dp &= \int_{p=0}^{p=2} \left(3p^2-4\right)^{1/3} [p]\, dp \\
&= \int_{u=-4}^{u=8} u^{1/3} \left[\frac{1}{6}\frac{du}{dp}\right] dp \\
&= \frac{1}{6}\int_{u=-4}^{u=8} u^{1/3} du \\
&= \frac{1}{6}\left(\frac{3u^{4/3}}{4}\right)\Bigg|_{u=-4}^{u=8} \\
&= \frac{1}{8}\left(8^{4/3} - (-4)^{4/3}\right) \\
&= \frac{1}{8}\left(16 - 4\sqrt[3]{4}\right)
\end{aligned}$$