REVIEW FOR EXAM III

Lorraine and Millwood are convinced that musical shoes are the wave of the future and have started making them. Unfortunately, Lorraine and Millwood cannot stand each other and have started rival corporations ShoeTunes and SoleMusic in 2012 to pursue their dream of filling the world with musical footwear.

1. ShoeTunes' yearly profit in 2012 was $\$80,000$, and Lorraine projects that annual profit will increase by an annual rate of 4.2%, compounded monthly, since the company was founded. Let P represent the annual profit in dollars for ShoeTunes, and let x represent *the number of years passed* since the company was founded.

   (a) What is the per-month percent change and the per-month growth rate?

   Solution. The per-month percent change will be $\frac{0.042}{12} \approx 0.0035$ (or 0.35%) and the per-month growth rate will be 1.0035.

   (b) What is the annual growth rate and the true annual percent change (APY)?

   Solution. The annual growth rate will be $(1.0035)^{12} \approx 1.04282$, and therefore the APY will be approximately 0.04282 (or 4.282%).

   (c) Let f be the function that gives P in terms of x. Write down the output formula for f using proper function notation.

   Solution. The output formula will be $P = f(x) \approx 80000 \cdot 1.0035^{12x}$ or $P = f(x) \approx 80000 \cdot 1.04282^x$.

   (d) What will ShoeTune's profit be three and one-half years after it was founded?

   Solution. The profit will be $f(3.5) \approx 80000 \cdot 1.04282^{3.5} \approx 92,645.16$.

   (e) How many years will it take for the company's profit to reach $\$200,000$?

   Solution. We need to solve the equation $200000 = f(x)$ for the unknown x. Now,

$$\begin{aligned} 200000 = f(x) &\implies 200000 \approx 80000 \cdot 1.04282^x \\ &\implies \frac{5}{2} \approx 1.04282^x \\ &\implies \ln\left(\frac{5}{2}\right) \approx x\ln(1.04282) \\ &\implies \frac{\ln(5/2)}{\ln(1.04282)} \approx x \\ &\implies 21.85 \text{ years } \approx x \end{aligned}$$

   It is worth noting that we do not want to get carried away with decimal places in our final answer. Since the compounding rate is *monthly*, the profit will remain constant from the end of each month until the end of the next. Our final answer will therefore only make sense to the nearest month. Now, 0.85 years is equivalent to ten months and six days. Since the profit is constant from the beginning to the end of the tenth month, it does not make sense to say that the profit will reach \$200,000 in 21 years, ten months, and six days. In reality, the profit will not reach \$200,000 until the end of the tenth month. Therefore, the best final answer would be to say that the profit will reach \$200,000 *after 21 years and ten months has passed.*

2. SoleMusic's annual profit in 2012 was also $\$80,000$. However its profit is expected to increase *continuously* by an annual rate of 4.2%. Let P represent the annual profit in dollars for SoleMusic, and let x represent *the number of years passed* since the company was founded.

(a) What is the annual growth factor and the true annual percent change (APY)?

Solution. Since we are dealing with continuous growth, the annual growth factor will be $e^{0.042} \approx 1.04289$, and the APY will therefore be approximately 0.04289 (or 4.289%).

(b) Let g be the function that gives P in terms of x. Write down the output formula for g using proper function notation.

Solution. The output formula will be $P = g(x) = 80000 \cdot \left(e^{0.042}\right)^x$ or $P = g(x) \approx 80000 \cdot 1.04289^x$.

(c) What will SoleMusic's profit be three and one-half years after it was founded?

Solution. The profit will be $g(3.5) \approx 80000 \cdot 1.04289^{3.5} \approx 92,666.92$.

(d) How many years will it take for the company's profit to reach \$200,000?

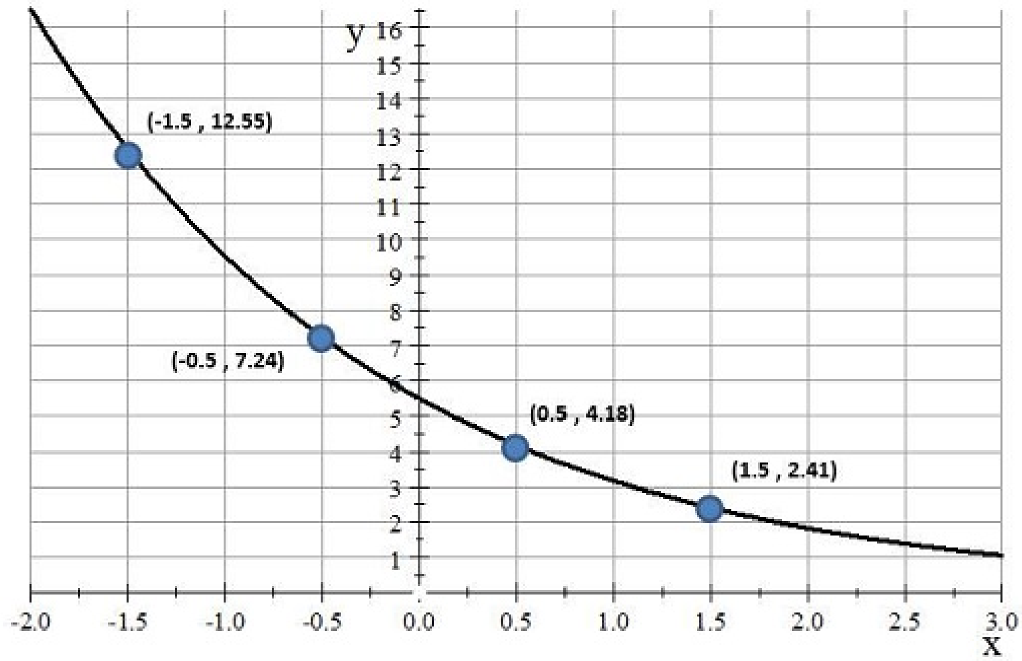
Solution. We need to solve the equation $200000 = g(x)$ for the unknown x. Now,

$$\begin{aligned}
200000 = g(x) \quad &\Longrightarrow \quad 200000 \approx 80000 \cdot 1.04289^x \\
&\Longrightarrow \quad \frac{5}{2} \approx 1.04289^x \\
&\Longrightarrow \quad \ln\left(\frac{5}{2}\right) \approx x \ln(1.04289) \\
&\Longrightarrow \quad \frac{\ln(5/2)}{\ln(1.04289)} \approx x \\
&\Longrightarrow \quad 21.81867 \text{ years } \approx x
\end{aligned}$$

Since growth is *continuous* in this case, we do not have to worry about time intervals where the profit will be constant. In this case, we used five decimal place accuracy in our final answer because that is the accuracy of our growth factor. (If we had used greater decimal place accuracy in the growth factor, we could have used greater accuracy in the final answer as well.) This answer would correspond to approximately 21 years, 9 months, 24 days, 17 hours, 27 minutes, and 10 seconds.

The graph below presents an exponential relationship between the quantities x and y. Let h be the name of the function that gives the values of y in terms of the values of x. Use this graph to answer

Questions 3 - 9.



3. According to the graph, what is the approximate value of $h(2.5)$?

   Solution. Based on the graph, $h(2.5) \approx 1.4$.

4. According to the graph, what is the approximate solution to the equation $10 = h(x)$?

   Solution. A horizontal line drawn from $y = 10$ crosses the graph at approximately $(-1.1, 10)$. Hence, $10 \approx h(-1.1)$.

5. To the nearest thousandth, what is the decay factor for the function h? What is the percent change?

   Solution. We can compute the decay factor using any pair of points on the graph, but the easiest approach is to use points where $\Delta x = 1$. For example, we can use the points $(-0.5, 7.24)$ and $(0.5, 4.18)$ or $(0.5, 4.18)$ and $(1.5, 2.41)$. When $\Delta x = 1$, the decay factor is simply the ratio of the output values. Thus,

$$\text{DECAY FACTOR } = \frac{4.18}{7.24} \approx 0.577 \quad \text{or} \quad \text{DECAY FACTOR } = \frac{2.41}{4.18} \approx 0.577$$

6. What is the initial value for the function h?

   Solution. The initial value of an exponential function occurs when the input value is 0. Therefore, if we have the graph of the function, the initial value is the second coordinate of the vertical intercept for the graph. Looking at the graph, we can see that the vertical intercept is approximately $(0, 5.5)$, so we know the initial value is approximately 5.5.

   Alternatively, we know that output formula for the function is going to be $y = h(x) \approx a \cdot (0.577)^x$. Since we know that $(1.5, 2.41)$ lies on the graph, we also know

$$2.41 \approx a \cdot (0.577)^{1.5} \Longrightarrow \frac{2.41}{(0.577)^{1.5}} \approx a \Longrightarrow 5.5 \approx a$$

   We could have used any point on the graph to find the value of a using this method.

7. Construct the output formula for the function h. Write your answer using proper function notation.

   Solution. The output formula will be $y = h(x) \approx 5.5 \cdot (0.577)^x$.

8. Use your formula to determine the approximate value of $h(2.5)$ and compare your answer to the one you obtained in Problem 3.

   Solution. Using the formula, we see that $h(2.5) \approx 5.5 \cdot (0.577)^{2.5} \approx 1.391$.

9. Use your formula to determine the approximate solution to the equation $10 = h(x)$ and compare your answer to the one you obtained in Problem 4.

   Solution. The equation $10 = h(x)$ tells us

$$
\begin{aligned}
10 \approx 5.5 \cdot (0.577)^x \quad &\Longrightarrow \quad 1.818 \approx 0.577^x \\
&\Longrightarrow \quad \ln(1.818) \approx x \ln(0.577) \\
&\Longrightarrow \quad \frac{\ln(1.818)}{\ln(0.577)} \approx x \\
&\Longrightarrow \quad -1.087 \approx x
\end{aligned}
$$

10. Rewrite each of the following logarithmic expressions as a single logarithm.

    (a) $\log_5(x) + 3\log_5(1-x)$ (b) $\dfrac{1}{2}\ln(x) - 2\ln(5)$ (c) $\log(x-3) + \log(x+3) - \log(4)$

    (a) We have $\log_5(x) + 3\log_5(1-x) = \log_3(x) + \log_5(1-x)^3 = \log_5\left[x\,(1-x)^3\right]$.

    (b) We have $\dfrac{1}{2}\ln(x) - 2\ln(5) = \ln(x^{1/2}) - \ln(5^2) = \ln\left[\dfrac{\sqrt{x}}{25}\right]$.

    (c) We have $\log(x-3) + \log(x+3) - \log(4) = \log\left[\dfrac{(x+3)(x-3)}{4}\right] = \log\left[\dfrac{x^2-9}{4}\right]$.

11. Solve the equation $\ln(x) + \ln(3x) - \ln(4) = 1$ for the unknown x.

    Solution. Using the laws of logarithms, we see that

$$
\begin{aligned}
\ln(x) + \ln(3x) - \ln(4) = 1 \quad &\Longrightarrow \quad \ln\left(\frac{3x^2}{4}\right) = 1 \\
&\Longrightarrow \quad \frac{3x^2}{4} = e^1 \\
&\Longrightarrow \quad x^2 = \frac{4e}{3} \\
&\Longrightarrow \quad x = \pm 2\sqrt{\frac{e}{3}}
\end{aligned}
$$

However, since $\ln\left(-2\sqrt{\dfrac{e}{3}}\right)$ is undefined, only $x = 2\sqrt{\dfrac{e}{3}}$ is a solution to the original equation.

12. Solve the equation $\log_2(y^2 + 3y - 6) = 2$ for the unknown y.

    Solution. Since only one logarithm appears in the equation, we do not need to use the laws of logarithms in this equation. We simply write the equation in exponential form and solve for x.

Observe

$$\begin{aligned}
\log_2(y^2+3y-6)=2 &\implies y^2+3y-6=2^2\\
&\implies y^2+3y-6=4\\
&\implies y^2+3y-10=0\\
&\implies (y+5)(y-2)=0\\
&\implies y=-5 \text{ or } y=2
\end{aligned}$$

Now, in this case, $\log_2\left[(-5)^2+3(-5)-6\right]=\log_2(4)$ is defined, and $\log_2\left[(2)^2+3(2)-6\right]=\log_2(4)$ is defined. Therefore, $y=-5$ and $y=2$ are both solutions to the original equation.