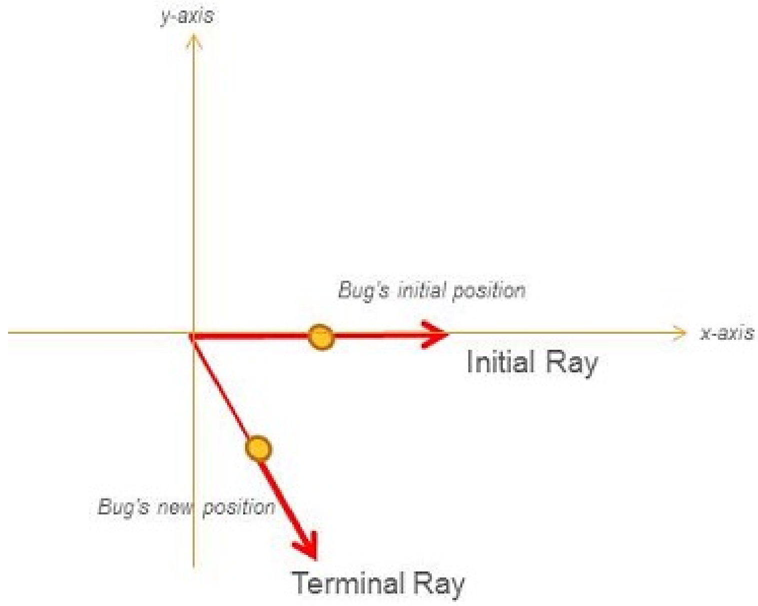
A Second Look at Angle Measure

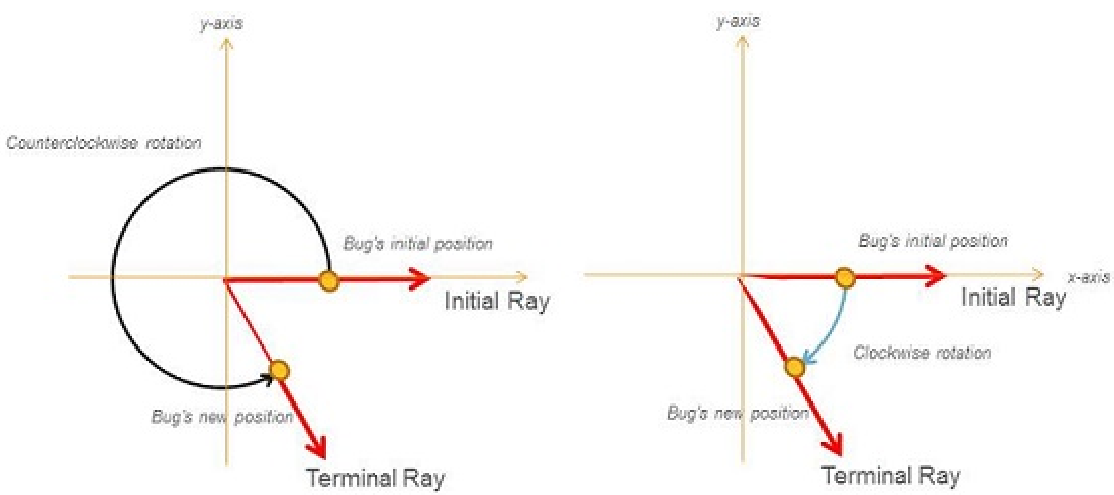
In the last two units, we introduced the cosine, sine, and tangent functions and sketched their graphs on the interval $0 \leq \theta \leq 2\pi$, where θ represents the radian measure of an angle formed by rotating in the counterclockwise direction from an initial point. It is time to take a closer look at angle measure, because there is more to it than meets the eye.

Remember that, when we first introduced angles as the "gap" between two rays having a common endpoint, we mentioned that these rays actually form *two* angles. Do these angles have the same radian measure? Let's think about this by returning to the bug problem.

Here is the diagram from Unit 2 showing the angle in standard position made by the bug as it rotates from its initial position to its new position, only this time with no direction arc to indicate which direction the bug traveled.



There are two ways the bug could have gotten from its initial position to its new position by traveling on the tip of the fan blade — the blade could have rotated in the *counterclockwise* direction, or it could have rotated in the *clockwise direction*. These two possibilities are illustrated in the figure below.



Compare the arcs cut by this angle from the two-foot circle when we rotate in the counterclockwise direction versus the clockwise direction. It is clear from the diagrams that these arcs have different lengths, and this means these arcs give rise to different radian measures for the angle.

There is more than one radian measure for an angle.

In order to tell the difference between the clockwise-oriented radian measure and the counterclockwise-oriented radian measure for an angle, we adopt the following convention:

Distinguishing Rotation Direction

Radian measure will be positive for counterclockwise rotation and negative for clockwise rotation.

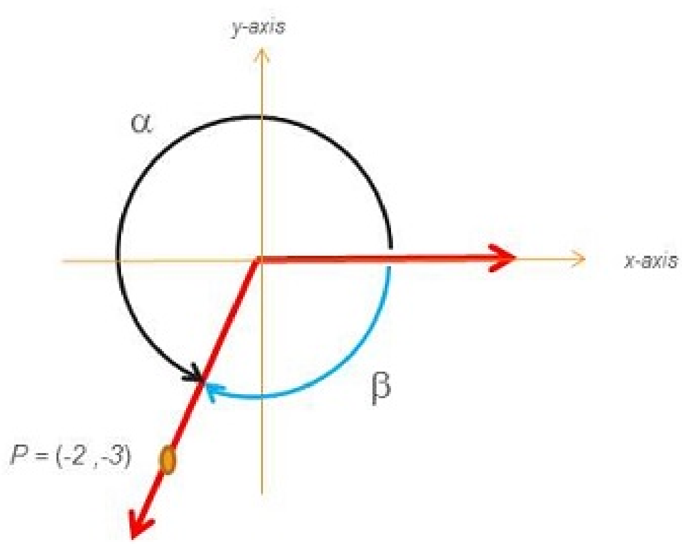
In other words, when you are told that an angle has radian measure -3.17 `rad`, this means that the angle represents a rotation through 3.17 radius lengths *in the clockwise direction* along the circumference of any circle centered on the angle vertex.

Problem 1. As the fan blade rotates, it traces out a circle with a two-foot radius. Suppose the bug rotates two-thirds of the way around this circle in the clockwise direction. What is the clockwise-oriented radian measure for this angle?

Problem 2. If the bug rotates two thirds of the way around the circle in the clockwise direction, how far around the circle in the *counterclockwise* direction would the bug have to rotate to form the same angle? What is the *counterclockwise*-oriented radian measure for this angle?

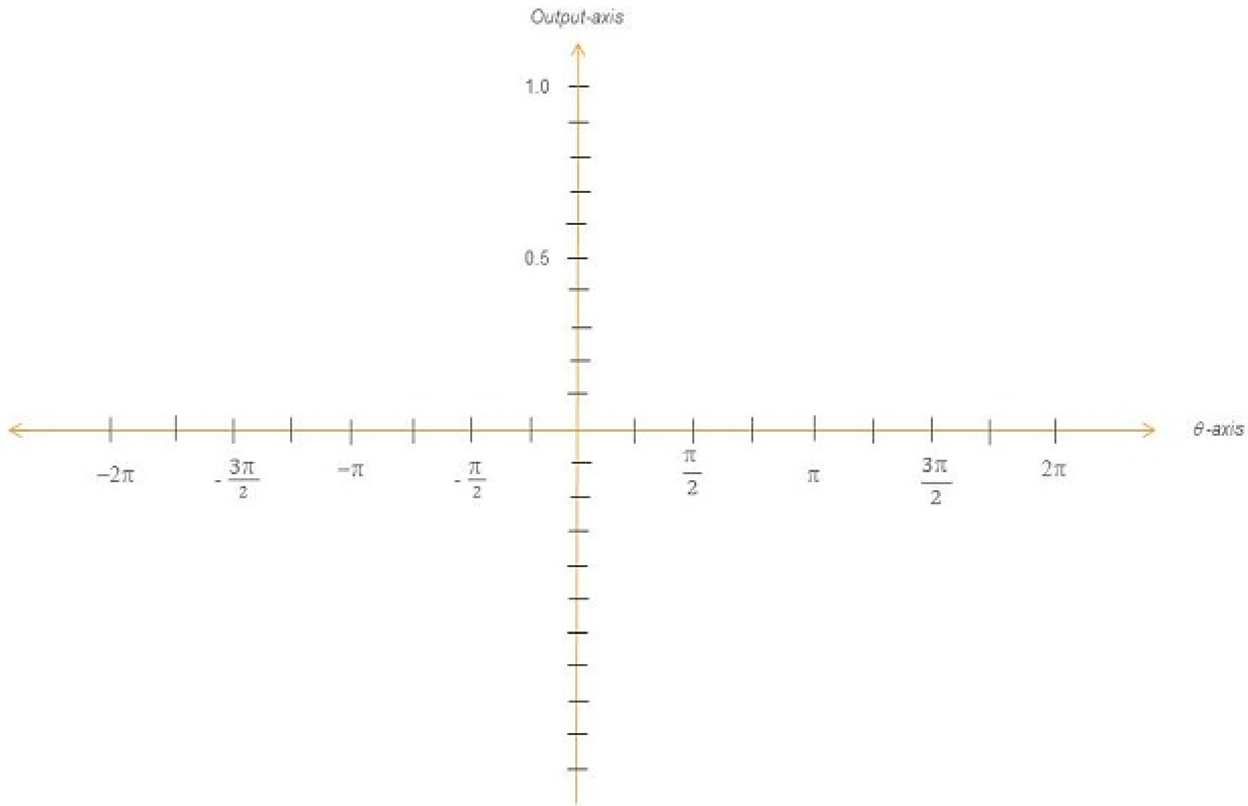
Problem 3. Suppose the bug rotates through an angle having radian measure $-\frac{5\pi}{8}$ `rad`. What is the *counterclockwise*-oriented radian measure for this angle?

Problem 4. Consider the angle in standard position shown below having counterclockwise-oriented radian measure α and clockwise-oriented radian measure β. Using the point P shown on the terminal side of this angle, what can you say about the relationship between $\sin(\alpha)$ and $\sin(\beta)$? What about the relationship between $\cos(\alpha)$ and $\cos(\beta)$?

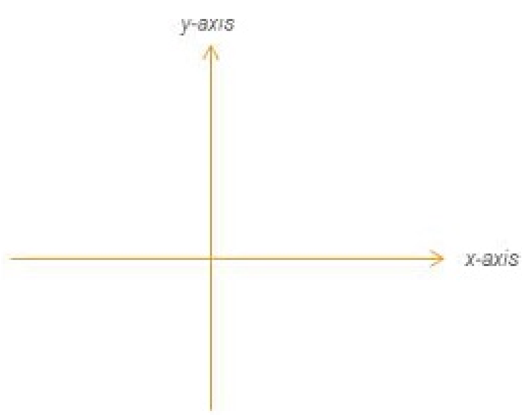


Problem 5. Suppose that α is the counterclockwise-oriented radian measure for an angle, and suppose that β is the clockwise-oriented radian measure for the same angle as shown in the diagram for Problem 4. Explain why we have $\alpha - \beta = 2\pi$ `rad`. (Remember, β is negative.)

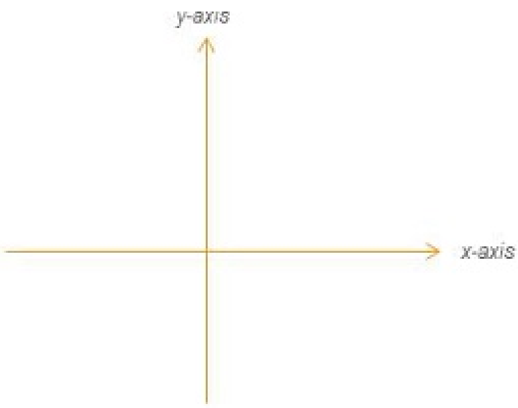
Problem 6. In the last unit, you sketched the graph of $f(\theta) = \cos(\theta)$ and $g(\theta) = \sin(\theta)$ on the interval $0 \leq \theta \leq 2\pi$. Using these graphs and what you learned from Problem 4 and 5, sketch the graphs of these functions on the interval $-2\pi \leq \theta \leq 2\pi$. Be sure to label your graphs.

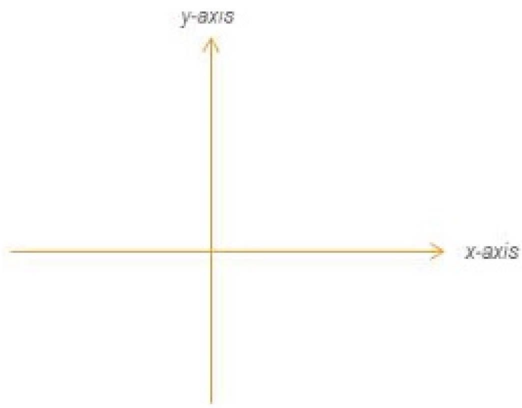


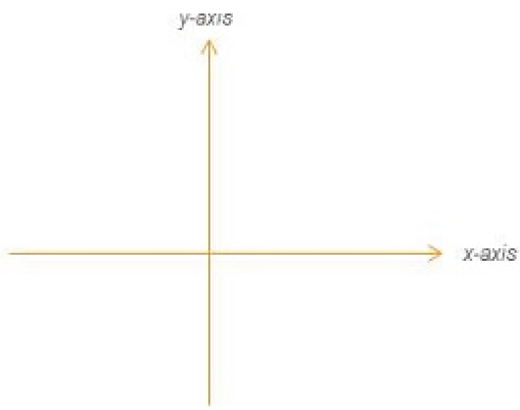
Problem 7. Let's return to the fan problem again. We know that an angle measure of 2π `rad` represents represents one complete rotation around the two-foot circle in the counterclockwise direction (so the bug's initial position and new position are the same). Suppose you are told that the bug has rotated through an angle having radian measure 3π `rad`. How would you explain the bug's motion around the fan? Draw this angle and its direction arc on the grid below.



Problem 8. On the first grid below, draw an angle in standard position that has radian measure $\frac{\pi}{4}$ `rad`. On the second grid, draw an angle in standard position having radian measure $-\frac{7\pi}{4}$ `rad`, and on the third grid, draw an angle in standard position that has radian measure $\frac{9\pi}{4}$ `rad`. Do you notice anything about these three angles?







Problem 9. Suppose that someone points out to you that

$$\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi \qquad -\frac{7\pi}{4} = \frac{\pi}{4} - 2\pi$$

How could you use this to explain what you noticed in Problem 9? Think in terms of rotations.

Problem 10. Use your calculator to approximate $\sin(3 \texttt{ rad})$, $\sin(3 + 2\pi \texttt{ rad})$, and $\sin(3 + 3\pi \texttt{ rad})$. How could you explain what you notice? Think in terms of the rotations involved in creating each angle from an initial ray on the positive x-axis.

The smallest positive radian measure for an angle is called the *fundamental measure* of that angle. All radian measures for an angle will differ from the fundamental measure by a series of complete rotations clockwise or counterclockwise. Therefore, all radian measures for an angle will differ from the funamental measure by a positive or a negative integer multiple of 2π `rad`.

Example 1 *What is the fundamental measure for the angle having radian measure* $\alpha = -\frac{23\pi}{8}$ *rad?*

Solution. The given measure is clockwise-oriented. To find the smallest positive measure for this angle, we simply *add* 2π `rad` to this measure repeatedly until we first get a positive number. Each value we get along the way will be another radian measure for this angle.

$$-\frac{23\pi}{8} + 2\pi = \frac{-23\pi + 16\pi}{8} = -\frac{7\pi}{8} \quad \text{A new radian measure for this angle}$$

$$-\frac{7\pi}{8} + 2\pi = \frac{-7\pi + 16\pi}{8} = \frac{9\pi}{8} \quad \text{A new radian measure for this angle}$$

We just went positive with the last addition of 2π `rad`, so $\theta = \frac{9\pi}{8}$ `rad` is the fundamental measure for this angle. Note that the original measure α differs from the fundamental measure by two complete *clockwise-oriented* rotations.

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Problem 11. Suppose the fundamental measure for an angle is $\theta = 3.50$ `rad`. The value $\alpha = 22.35$ `rad` is another radian measure for this angle. By how many complete rotations (and in what direction) does α differ from θ?

Problem 12. What is the fundamental measure for the angle having radian measure $\alpha = \frac{43\pi}{6}$ `rad`? By how many complete rotations (and in what direction) does α differ from the fundamental measure?

Supplemental Problems

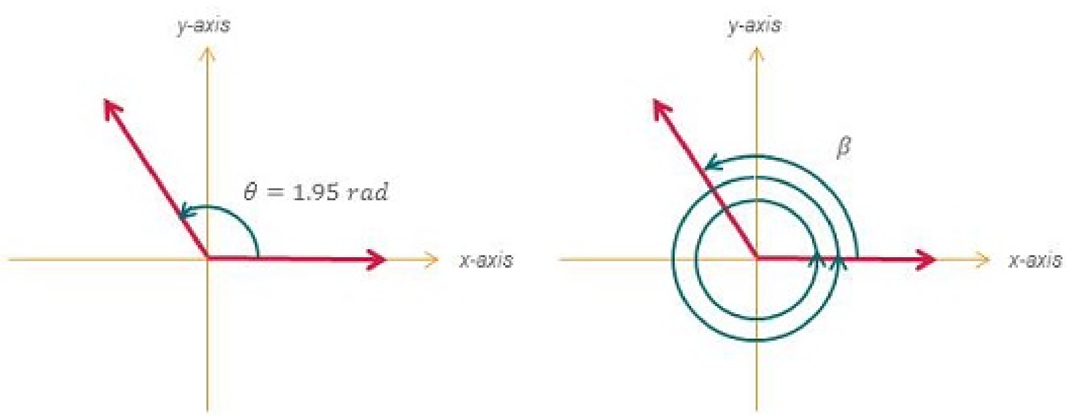
1. What would be a *clockwise-oriented* measure for the angles whose radian measure is given below?

   (a) $\theta = \frac{4\pi}{3}$ `rad` (b) $\theta = \frac{8\pi}{5}$ `rad` (c) $\theta = 2.45$ `rad`

2. What would be a *counterclockwise-oriented* measure for the angles whose radian measure is given below?

   (a) $\theta = -\frac{4\pi}{3}$ `rad` (b) $\theta = -\frac{\pi}{2}$ `rad` (c) $\theta = -1.73$ `rad`

3. The diagram below shows two different radian measures for an angle in standard position. If $\theta = 1.95$ `rad`, what is the value of β? (The value of β must include the complete rotations shown.)



4. Suppose the fundamental measure for an angle is $\theta = 1.88$ `rad`. Another measure β for this angle is obtained by adding seven complete clockwise-oriented rotations to the value of θ. What is the radian measure of β?

5. Determine the fundamental measure for the angles having the following radian measures. By how many complete rotations (and in what direction) do these measures differ from the fundamental measure?

   (a) $\beta = \dfrac{23\pi}{7}$ `rad` (b) $\alpha = -\dfrac{33\pi}{5}$ `rad` (c) $\delta = -28.91$ `rad`

6. Explain why the following statement is true. "If α and β are radian measures for the same angle in standard position, then $\sin(\alpha) = \sin(\beta)$."

7. Is the following statement true? Explain your reasoning. "If α and β differ by 3π then $\cos(\alpha) = \cos(\beta)$."

8. Is the following statement true? Explain your reasoning. "If θ is the fundamental measure for an angle in standard position, then $\tan(\theta) = \tan(\theta - 12\pi)$."