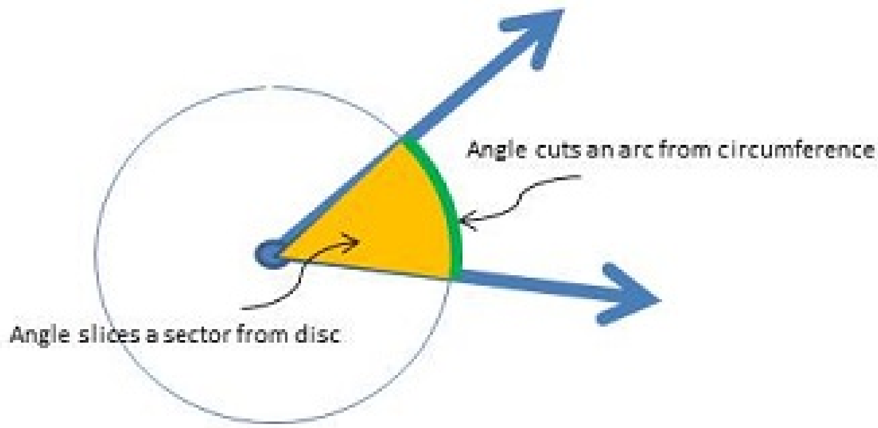
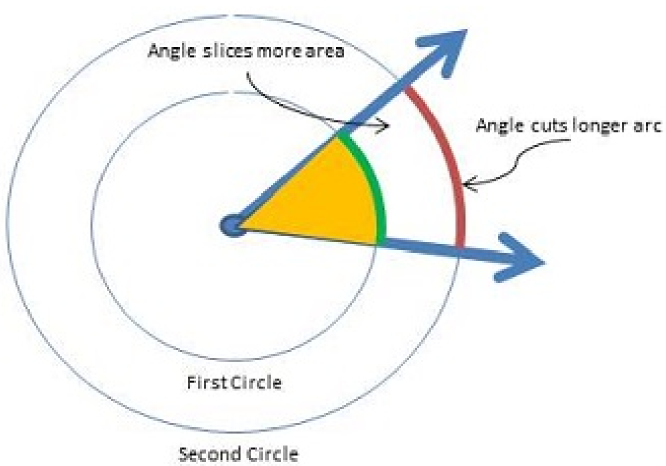
Measuring Angles

An *angle* is the "gap" formed by two rays that have the same endpoint. This endpoint is called the *vertex* of the angle. Every pair of rays that have the same endpoint actually make *two* angles, as the diagram below shows.

We have two standard ways to measure angles, and to understand something about these ways, it is helpful to let the vertex of your angle be the center of a circle. It does not matter what the radius of this circle is.

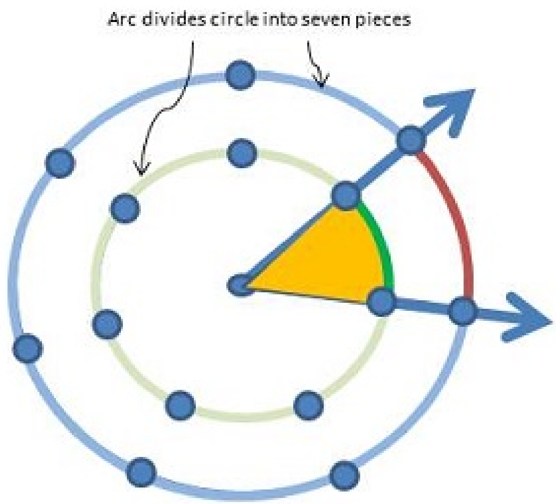


The angle shown above slices a sector from the interior of the circle, and it also cuts an arc from the circumference of the circle. We could measure either of these attributes for the angle, so we might consider using either one as a way to measure the angle. However, there is a big problem with this approach — what if we use a *different* circle instead?



As the figure above shows, if we use a bigger circle, the *same* angle slices a larger sector and cuts a longer arc. For this reason, we don't use attributes that come directly from a circle to measure angles. We adopt a more subtle approach.

First, notice that if we mark off the arcs cut by the angle around their circles, we can see that they each divide their circles into the same number of pieces.



Although the arcs cut by this angle from each circle have different lengths, they are both one-seventh the circumference of the circles they are cut from. This observation points out something critical about the relationship between the circles we draw and the arcs cut by the angle from each one:

Fundamental Property of Angles and Arcs

- If the vertex of an angle is at the center of two circles, the arcs cut by the angle will be the same fraction of their circle's circumference.

Suppose we let R be the radius of the outer circle in the diagram above (measured in inches, let's say). The circumference of the circle is a constant multiple of the radius; in fact, the circumference is $C = 2\pi R$ inches. This means that the length S of the arc cut by this angle from the circle is also some multiple of R. What mulitple is it? We know that S is one-seventh the circumference, so

$$S = \frac{1}{7} \cdot 2\pi R \text{ inches} \quad \Longrightarrow \quad S = \left(\frac{2\pi}{7}\right) \cdot R \text{ inches}$$

Since both S and R are measured in inches, the fraction $\frac{2\pi}{7}$ has no units associated with it. We call this unitless number the *radian measure* of the angle shown in the diagram above.

RADIAN MEASURE

- Suppose the vertex of an angle is at the center of a circle of radius R, measured in whatever units of length you wish. The angle will cut an arc of length S from the circumference of this circle (measured in the same units as R). The *radian measure* of this angle is the unitless constant θ we must mulitply R by in order to get the length S. In other words, we have the following three relationships that define the radian measure of this angle.

$$S = \theta \cdot R \qquad \theta = \frac{S}{R} \qquad R = \frac{S}{\theta}$$

Problem 1. Suppose the vertex of an angle is the center of a circle having a two foot radius. If the arc cut from this circle by the angle is five inches long, what is the radian measure of this angle?

Problem 2. Suppose the radian measure of an angle is $\theta = 2.31$. If the vertex of this angle is the center of a circle with a three-meter radius, how long is the arc cut by this angle?

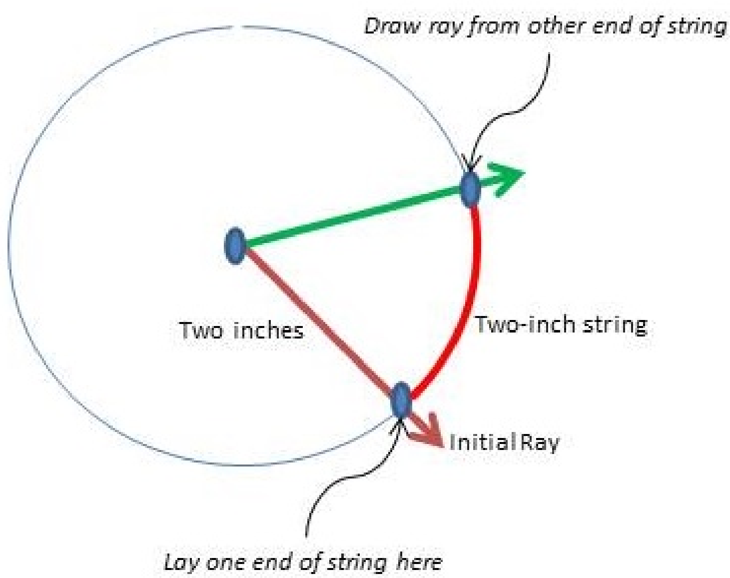
Problem 3. Suppose the radian measure of an angle is $\theta = \dfrac{7\pi}{8}$. If the vertex of this angle is the center of a circle, and the angle cuts an arc three feet long from this circle, what is the length of the radius?

Problem 4. The vertex of an angle is at the center of a circle whose radius is measured in miles, and the arc it cuts from this circle is exactly one-fourth the circle's circumference. What is the radian measure of this angle?

The radian measure of an angle is unitless. However, it is sometimes helpful to have some type of unit to work with when we are dealing with radian measure, so we invent a special unit just for this purpose. One *radian* is defined to be the angle required to cut an arc from any circle that is exactly equal in length to the circle's radius. This special angle is called a `rad` — the arc cut from any circle by a `rad` is equal to one radius length. The radian measure of any angle can be found by counting the number of `rad`'s enclosed between the two rays that define the angle.

Example 1 *What does a radian look like?*

Solution. A radian is a special angle. To see what it looks like, let's work with a circle whose radius is two inches. When its vertex is the center of this circle, a radian will cut an arc from the circle that is exactly two inches long. We can draw one radius for this circle, then cut a two-inch piece of string and lay it on the perimeter of the circle so that one end lies at point where the radius we drew intersects the circle. We then draw another ray from the other end of the string back to the center. The resulting "gap"is one radian.



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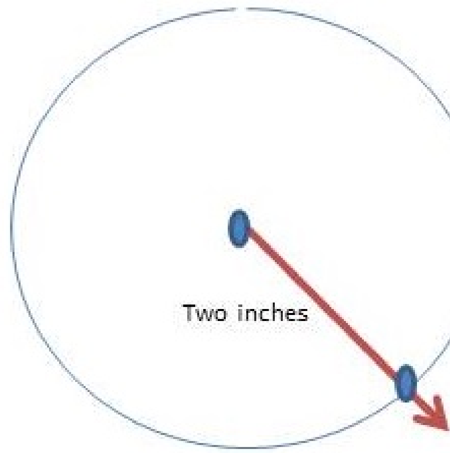
Problem 5. What does it mean to say that an angle has a measure of 2.67 `rad`?

Problem 6. Assume the circle below has a radius of two inches. Using the initial ray given, draw an angle whose measure is exactly 3 `rad`.



DEGREE MEASURE

Degree measure is another, much older way to measure angles. It is not used much in mathematics, physics, or engineering for a number of reasons, most of which have to do with calculus. It is still widely used in navigation, surveying, and meteorology however; and it is therefore worth mentioning. In degree measure, a circle is is divided into 360 equal sectors called *degrees*. To measure an angle in degrees, we imagine its vertex is at the center of a circle (of any radius), and we simply count the number of degrees enclosed between the rays that define the angle. It is traditional to use a superscipted circle on the degree measurement to indicate that the angle measurement is in degrees. For example

Angle measure of 38.92 degrees is written $38.92°$

Example 2 *What is the radian measure of an angle whose degree measure is* $120°$*?*

Solution. Suppose the vertex of our angle is the center of a circle of radius R (measured in inches, let's say). To answer this question, we first observe that 120 degrees is

$$\frac{120}{360} = \frac{1}{3}$$

the total degree measure of the circle. Consequently, our angle will cut an arc that is one-third the total circumference of the circle. The length of this arc will therefore be $S = \frac{1}{3} \cdot (2\pi R)$ inches. The radian measure of this angle is

$$\theta = \frac{2\pi}{3}\ \texttt{rad}$$

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Problem 7. Use the method of the previous example to fill in the table below. Assume the vertex of each angle is the center of a circle of radius R inches.

Degree Measure of Angle	Fraction of 360°	Length of Arc	Radian Measure
0°			
15°			
30°			

Problem 8. On the grid provided, plot the radian measure of the angles above as a function of their degree measure. What do you notice about these points?

Problem 9. Let d be the degree measure of an angle, and let θ be its radian measure. Use the table above to construct a function f that gives θ in terms of d.

Problem 10. Let g be the function that reverses f. Find a formula for the function g.

Problem 11. What is the meaning of the equation $\theta = f(d)$? What is the meaning of the equation $d = g(\theta)$?

Problem 12. Use your work in Problems 9, 10, and 11 to explain why the following conversion formulas work.

- If d is the degree measure of an angle, and θ is the radian measure of the same angle, then

$$\theta = \frac{\pi}{180} \cdot d \qquad\qquad d = \frac{180}{\pi} \cdot \theta$$

Supplemental Homework Problems

Exercises 7, 8, 12, 13, 14, and 15 from Module 7 pp. 299 - 301