

Chapter 3

Review of Statistics

3.1. The central limit theorem suggests that when the sample size (n) is large, the distribution of the sample average (\bar{Y}) is approximately $N(\mu_Y, \sigma_Y^2)$ with $\sigma_Y^2 = \frac{\sigma_Y^2}{n}$. Given a population $\mu_Y = 100$, $\sigma_Y^2 = 43.0$, we have

(a) $n = 100$, $\sigma_Y^2 = \frac{\sigma_Y^2}{n} = \frac{43}{100} = 0.43$, and

$$\Pr(\bar{Y} < 101) = \Pr\left(\frac{\bar{Y} - 100}{\sqrt{0.43}} < \frac{101 - 100}{\sqrt{0.43}}\right) \approx \Phi(1.525) = 0.9364.$$

(b) $n = 64$, $\sigma_Y^2 = \frac{\sigma_Y^2}{n} = \frac{43}{64} = 0.6719$, and

$$\begin{aligned} \Pr(101 < \bar{Y} < 103) &= \Pr\left(\frac{101 - 100}{\sqrt{0.6719}} < \frac{\bar{Y} - 100}{\sqrt{0.6719}} < \frac{103 - 100}{\sqrt{0.6719}}\right) \\ &\approx \Phi(3.6599) - \Phi(1.2200) = 0.9999 - 0.8888 = 0.1111. \end{aligned}$$

(c) $n = 165$, $\sigma_Y^2 = \frac{\sigma_Y^2}{n} = \frac{43}{165} = 0.2606$, and

$$\begin{aligned} \Pr(\bar{Y} > 98) &= 1 - \Pr(\bar{Y} \leq 98) = 1 - \Pr\left(\frac{\bar{Y} - 100}{\sqrt{0.2606}} \leq \frac{98 - 100}{\sqrt{0.2606}}\right) \\ &\approx 1 - \Phi(-3.9178) = \Phi(3.9178) = 1.0000 \text{ (rounded to four decimal places).} \end{aligned}$$

3.2. Each random draw Y_i from the Bernoulli distribution takes a value of either zero or one with probability $\Pr(Y_i = 1) = p$ and $\Pr(Y_i = 0) = 1 - p$. The random variable Y_i has mean

$$E(Y_i) = 0 \times \Pr(Y = 0) + 1 \times \Pr(Y = 1) = p,$$

and variance

$$\begin{aligned} \text{var}(Y_i) &= E[(Y_i - \mu_Y)^2] \\ &= (0 - p)^2 \times \Pr(Y_i = 0) + (1 - p)^2 \times \Pr(Y_i = 1) \\ &= p^2(1 - p) + (1 - p)^2 p = p(1 - p). \end{aligned}$$

- (a) The fraction of successes is

$$\hat{p} = \frac{\#(\text{success})}{n} = \frac{\#(Y_i = 1)}{n} = \frac{\sum_{i=1}^n Y_i}{n} = \bar{Y}.$$

$$(b) \quad E(\hat{p}) = E\left(\frac{\sum_{i=1}^n Y_i}{n}\right) = \frac{1}{n} \sum_{i=1}^n E(Y_i) = \frac{1}{n} \sum_{i=1}^n p = p.$$

$$(c) \quad \text{var}(\hat{p}) = \text{var}\left(\frac{\sum_{i=1}^n Y_i}{n}\right) = \frac{1}{n^2} \sum_{i=1}^n \text{var}(Y_i) = \frac{1}{n^2} \sum_{i=1}^n p(1-p) = \frac{p(1-p)}{n}.$$

The second equality uses the fact that Y_1, \dots, Y_n are i.i.d. draws and $\text{cov}(Y_i, Y_j) = 0$, for $i \neq j$.

- 3.3. Denote each voter's preference by Y . $Y = 1$ if the voter prefers the incumbent and $Y = 0$ if the voter prefers the challenger. Y is a Bernoulli random variable with probability $\Pr(Y = 1) = p$ and $\Pr(Y = 0) = 1 - p$. From the solution to Exercise 3.2, Y has mean p and variance $p(1 - p)$.

$$(a) \quad \hat{p} = \frac{215}{400} = 0.5375.$$

- (b) The estimated variance of \hat{p} is $\text{var}(\hat{p}) = \frac{\hat{p}(1 - \hat{p})}{n} = \frac{0.5375 \times (1 - 0.5375)}{400} = 6.2148 \times 10^{-4}$. The standard error is $\text{SE}(\hat{p}) = (\text{var}(\hat{p}))^{\frac{1}{2}} = 0.0249$.

- (c) The computed t -statistic is

$$t^{act} = \frac{\hat{p} - \mu_{p,0}}{\text{SE}(\hat{p})} = \frac{0.5375 - 0.5}{0.0249} = 1.506.$$

Because of the large sample size ($n = 400$), we can use Equation (3.14) in the text to get the p -value for the test $H_0 : p = 0.5$ vs. $H_1 : p \neq 0.5$:

$$p\text{-value} = 2\Phi(-|t^{act}|) = 2\Phi(-1.506) = 2 \times 0.066 = 0.132.$$

- (d) Using Equation (3.17) in the text, the p -value for the test $H_0 : p = 0.5$ vs. $H_1 : p > 0.5$ is

$$p\text{-value} = 1 - \Phi(t^{act}) = 1 - \Phi(1.506) = 1 - 0.934 = 0.066.$$

- (e) Part (c) is a two-sided test and the p -value is the area in the tails of the standard normal distribution outside \pm (calculated t -statistic). Part (d) is a one-sided test and the p -value is the area under the standard normal distribution to the right of the calculated t -statistic.
- (f) For the test $H_0 : p = 0.5$ vs. $H_1 : p > 0.5$, we cannot reject the null hypothesis at the 5% significance level. The p -value 0.066 is larger than 0.05. Equivalently the calculated t -statistic 1.506 is less than the critical value 1.64 for a one-sided test with a 5% significance level. The test suggests that the survey did not contain statistically significant evidence that the incumbent was ahead of the challenger at the time of the survey.

- 3.4. Using Key Concept 3.7 in the text

- (a) 95% confidence interval for p is

$$\hat{p} \pm 1.96SE(\hat{p}) = 0.5375 \pm 1.96 \times 0.0249 = (0.4887, 0.5863).$$

- (b) 99% confidence interval for p is

$$\hat{p} \pm 2.57SE(\hat{p}) = 0.5375 \pm 2.57 \times 0.0249 = (0.4735, 0.6015).$$

- (c) Mechanically, the interval in (b) is wider because of a larger critical value (2.57 versus 1.96). Substantively, a 99% confidence interval is wider than a 95% confidence because a 99% confidence interval must contain the true value of p in 99% of all possible samples, while a 95% confidence interval must contain the true value of p in only 95% of all possible samples.
- (d) Since 0.50 lies inside the 95% confidence interval for p , we cannot reject the null hypothesis at a 5% significance level.

- 3.5. (a) (i) The size is given by $\Pr(|\hat{p} - 0.5| > .02)$, where the probability is computed assuming that $p = 0.5$.

$$\begin{aligned} \Pr(|\hat{p} - 0.5| > 0.02) &= 1 - \Pr(-0.02 \leq \hat{p} - 0.5 \leq .02) \\ &= 1 - \Pr\left(\frac{-0.02}{\sqrt{0.5 \times 0.5/1055}} \leq \frac{\hat{p} - 0.5}{\sqrt{0.5 \times 0.5/1055}} \leq \frac{0.02}{\sqrt{0.5 \times 0.5/1055}}\right) \\ &= 1 - \Pr\left(-1.30 \leq \frac{\hat{p} - 0.5}{\sqrt{0.5 \times 0.5/1055}} \leq 1.30\right) \\ &= 0.19 \end{aligned}$$

where the final equality using the central limit theorem approximation.

- (ii) The power is given by $\Pr(|\hat{p} - 0.5| > 0.02)$, where the probability is computed assuming that $p = 0.53$.

$$\begin{aligned} \Pr(|\hat{p} - 0.5| > 0.02) &= 1 - \Pr(-0.02 \leq \hat{p} - 0.5 \leq .02) \\ &= 1 - \Pr\left(\frac{-0.02}{\sqrt{0.53 \times 0.47/1055}} \leq \frac{\hat{p} - 0.5}{\sqrt{0.53 \times 0.47/1055}} \leq \frac{0.02}{\sqrt{0.53 \times 0.47/1055}}\right) \\ &= 1 - \Pr\left(\frac{-0.05}{\sqrt{0.53 \times 0.47/1055}} \leq \frac{\hat{p} - 0.53}{\sqrt{0.53 \times 0.47/1055}} \leq \frac{-0.01}{\sqrt{0.53 \times 0.47/1055}}\right) \\ &= 1 - \Pr\left(-3.25 \leq \frac{\hat{p} - 0.53}{\sqrt{.53 \times 0.47/1055}} \leq -0.65\right) \\ &= 0.74 \end{aligned}$$

where the final equality using the central limit theorem approximation.

- (b) (i) $t = \frac{0.54 - 0.50}{\sqrt{(0.54 \times 0.46)/1055}} = 2.61$, and $\Pr(|t| > 2.61) = 0.01$, so that the null is rejected at the 5% level.
- (ii) $\Pr(t > 2.61) = .004$, so that the null is rejected at the 5% level.
- (iii) $0.54 \pm 1.96 \sqrt{(0.54 \times 0.46)/1055} = 0.54 \pm 0.03$, or 0.51 to 0.57.
- (iv) $0.54 \pm 2.58 \sqrt{(0.54 \times 0.46)/1055} = 0.54 \pm 0.04$, or 0.50 to 0.58.
- (v) $0.54 \pm 0.67 \sqrt{(0.54 \times 0.46)/1055} = 0.54 \pm 0.01$, or 0.53 to 0.55.

- (c) (i) The probability is 0.95 in any single survey, there are 20 independent surveys, so the probability is $0.95^{20} = 0.36$
- (ii) 95% of the 20 confidence intervals or 19.
- (d) The relevant equation is $1.96 \times \text{SE}(\hat{p}) < 0.01$ or $1.96 \times \sqrt{p(1-p)/n} < 0.01$. Thus n must be chosen so that $n > \frac{1.96^2 p(1-p)}{0.01^2}$, so that the answer depends on the value of p . Note that the largest value that $p(1-p)$ can take on is 0.25 (that is, $p = 0.5$ makes $p(1-p)$ as large as possible). Thus if $n > \frac{1.96^2 \times 0.25}{0.01^2} = 9604$, then the margin of error is less than 0.01 for all values of p .
- 3.8 $1110 \pm 1.96 \left(\frac{123}{\sqrt{1000}} \right)$ or 1110 ± 7.62 .
- 3.9. Denote the life of a light bulb from the new process by Y . The mean of Y is μ and the standard deviation of Y is $\sigma_Y = 200$ hours. \bar{Y} is the sample mean with a sample size $n = 100$. The standard deviation of the sampling distribution of \bar{Y} is $\sigma_{\bar{Y}} = \frac{\sigma_Y}{\sqrt{n}} = \frac{200}{\sqrt{100}} = 20$ hours. The hypothesis test is $H_0: \mu = 2000$ vs. $H_1: \mu > 2000$. The manager will accept the alternative hypothesis if $\bar{Y} > 2100$ hours.

- (a) The size of a test is the probability of erroneously rejecting a null hypothesis when it is valid.

The size of the manager's test is

$$\begin{aligned} \text{size} &= \Pr(\bar{Y} > 2100 | \mu = 2000) = 1 - \Pr(\bar{Y} \leq 2100 | \mu = 2000) \\ &= 1 - \Pr\left(\frac{\bar{Y} - 2000}{20} \leq \frac{2100 - 2000}{20} | \mu = 2000\right) \\ &= 1 - \Phi(5) = 1 - 0.999999713 = 2.87 \times 10^{-7}, \end{aligned}$$

where $\Pr(\bar{Y} > 2100 | \mu = 2000)$ means the probability that the sample mean is greater than 2100 hours when the new process has a mean of 2000 hours.

- (b) The power of a test is the probability of correctly rejecting a null hypothesis when it is invalid. We calculate first the probability of the manager erroneously accepting the null hypothesis when it is invalid:

$$\begin{aligned} \beta &= \Pr(\bar{Y} \leq 2100 | \mu = 2150) = \Pr\left(\frac{\bar{Y} - 2150}{20} \leq \frac{2100 - 2150}{20} | \mu = 2150\right) \\ &= \Phi(-2.5) = 1 - \Phi(2.5) = 1 - 0.9938 = 0.0062. \end{aligned}$$

The power of the manager's testing is $1 - \beta = 1 - 0.0062 = 0.9938$.

- (c) For a test with 5%, the rejection region for the null hypothesis contains those values of the t -statistic exceeding 1.645.

$$t^{act} = \frac{\bar{Y}^{act} - 2000}{20} > 1.645 \Rightarrow \bar{Y}^{act} > 2000 + 1.645 \times 20 = 2032.9.$$

The manager should believe the inventor's claim if the sample mean life of the new product is greater than 2032.9 hours if she wants the size of the test to be 5%.

- 3.12. Sample size for men $n_1 = 100$, sample average $\bar{Y}_1 = 3100$ sample standard deviation $s_1 = 200$.
Sample size for women $n_2 = 64$, sample average $\bar{Y}_2 = 2900$, sample standard deviation $s_2 = 320$.

The standard error of $\bar{Y}_1 - \bar{Y}_2$ is $SE(\bar{Y}_1 - \bar{Y}_2) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{200^2}{100} + \frac{320^2}{64}} = 44.721$.

- (a) The hypothesis test for the difference in mean monthly salaries is

$$H_0: \mu_1 - \mu_2 = 0 \quad \text{vs.} \quad H_1: \mu_1 - \mu_2 \neq 0.$$

The t -statistic for testing the null hypothesis is

$$t^{act} = \frac{\bar{Y}_1 - \bar{Y}_2}{SE(\bar{Y}_1 - \bar{Y}_2)} = \frac{3100 - 2900}{44.721} = 4.4722.$$

Use Equation (3.14) in the text to get the p -value:

$$p\text{-value} = 2\Phi(-|t^{act}|) = 2\Phi(-4.4722) = 2 \times (3.8744 \times 10^{-6}) = 7.7488 \times 10^{-6}.$$

The extremely low level of p -value implies that the difference in the monthly salaries for men and women is statistically significant. We can reject the null hypothesis with a high degree of confidence.

- (b) From part (a), there is overwhelming statistical evidence that mean earnings for men *differ* from mean earnings for women, and a related calculation shows overwhelming evidence that mean earning for men are *greater* than mean earnings for women. However, by itself, this does not imply gender discrimination by the firm. Gender discrimination means that two workers, identical in every way but gender, are paid different wages. The data description suggests that some care has been taken to make sure that workers with similar jobs are being compared. But, it is also important to control for characteristics of the workers that may affect their productivity (education, years of experience, etc.). If these characteristics are systematically different between men and women, then they may be responsible for the difference in mean wages. (If this is true, it raises an interesting and important question of why women tend to have less education or less experience than men, but that is a question about something other than gender discrimination by this firm.) Since these characteristics are not controlled for in the statistical analysis, it is premature to reach a conclusion about gender discrimination.