Remote Learning Module for 8 April 2020

Lecture Notes on *Mind, Matter, and Mathematics* – Chapter 3

Last class we met the neurobiologist, Jean-Pierre Changeux, and the mathematician, Alain Connes, with our focus trained on their disputes over the nature of mathematical objects and how best to account for our knowledge of mathematical truths. We saw that Connes takes up the mantle of classical Platonism as regards the ontological status of mathematical objects—a view also known as Realism, and sometimes as Logicism. Changeux we found defends the contrary view called Constructionism, noting that this anti-realist view about mathematical truths comes in two distinct flavors, Intuitionism and Formalism (one of the topics we'll take up today, as we consider Chapter 3: "Nature Made to Order."

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(1) Constructivist Mathematics.

As regards the ontological status of mathematical objects, there are actually three quite distinct accounts, each serving as the ultimate foundations of mathematical knowledge. Now the view adopted by Alain Connes, Platonic Realism about mathematical entities, is of a piece with the foundational thinking of Gottlieb Frege, Bertrand Russell & A. N. Whitehead, Alonzo Church, Rudolf Carnap, and Kurt Gödel. This view is sometimes called just plain Realism, and sometimes Logicism. According to this view, abstract mathematical entities exist independently of our claims about them; things like sets and groups may or may not be discovered by our investigative efforts, but either way, they are quite real.

There are two strands of anti-realism about abstract entities: *Intuitionism* (as in the foundations of arithmetic and geometry devised by Poincaré, in France, and Brower and Heytig, in Holland). On this view, mathematical objects have the same ontological status as ordinary concepts—they exist in our minds, not independently of our understanding and intuitions (hence the name). Poincaré put this quite succinctly: "In math, to exist means to be free from contradiction." The other strand of constructionism (or anti-realism) comes from the work of David Hilbert; it's known as *Formalism*. On Hilbert's view, mathematical entities are literally (and please note that we are using the word, "literally" literally here) nothing but strings of symbols (and/or sounds) which have no content on their own—these symbols are introduced by way of logic in various uninterpreted systems, regulated by transformation rules (very much like the way we think of computer code as it is processed by a digital machine).

It is worth our further noting that these three accounts of the foundations of mathematics have a long ancestry in the old classical and mediaeval problem about the ontological status of universals (things like Humanity, Beauty, Goodness, and Truth). To be sure, Plato was the great-grandfather of Realism, holding in his Theory of Forms that universals (or types) are the originals of which particular things are copies (or tokens). In Plato's day, this theory of his stood in contrast with the earlier views of Heraclitus and Cratylus, who supposed that terms like Humanity or Beauty are things in name only—short-cuts for avoiding long lists of names for individual people or artifacts; this view became known as Nominalism. Aristotle also denied the reality of Plato's forms, although his view would become known as Conceptualism, because Aristotle figured there were such things as the idea of humanity and the idea of beauty in general, and that these general things existed in our minds in the form of concepts acquired by the intellectual agency of generalization. All this is to say that the dispute we find between Changeux and Connes is of ancient lineage.

Classical Universals	*Foundations of Mathematics*
Nominalism	Formalism
Conceptualism	Intuitionism
Realism	Logicism

One further note about Intuitionism: for Poincaré *et al*. there are no evidence-transcendent truths whatsoever. He once asked, rhetorically: What if last night, while the whole world was sleeping, the entire universe and everything in it doubled in size; would we have reason to say so? His answer was, No. There would be no reason, no evidence for saying that everything trebled in size or quadrupled in size either, because everything would look just the same as it does now. In other words, Nocturnal Doubling is an empty idea. He was wrong about his example, though; if you think about the effect of gravitation on pendulum clocks as opposed to the effect of tension on spring wound clocks, you'll see why (they'd no longer be in sync had Nocturnal Doubling occurred; remember, gravitation is an inverse square law). But you get his point: if there can, in principle, be no evidence for a claim, it is nonsense to assert it. Now, in mathematics, evidence amounts to formal proof, and so, for the Intuitionists, logic is trivalent, not bivalent. Instead of regarding mathematical statements as being either true or false, the Intuitionists regard statements as taking one of three values: provable, provably unprovable, and unproven. What classical mathematics takes for a true statement, intuitionist mathematics takes are having been proven; and what classical thinking takes for a false statement, intuitionism takes as provably unprovable. This may not seem like much of a difference until you notice that in classical logic the law of excluded middle holds for all declarative sentences: every assertion is either true or false, so that there is no *tertium quid* between the two values, true/false. But for intuitionist logic the law of excluded middle does not hold: there is a *tertium quid*, unproven, that stands between the provable and the provably unprovable.

Gödel famously countered Poincaré in his first Incompleteness Proof, which demonstrates of arithmetic that there are more truths than there are theorems. We should note that this proof demands a never ending scale of higher and higher orders of infinity (only the first of which shows that there are more real numbers than counting numbers), an ontology that Intuitionists reject in the first place. Given their tri-valent logic, they were, or course, quite right to do so, because all species of Indirect Proof (which we need in order to postulate higher order infinities) depend on the Law of Excluded Middle, and this law does not hold in Intuitionist logic.

(2) Kant.

These foundational issues also come into the foreground in Kant's account of *synthetic apriori* judgments. Before Kant, it was assumed that every truth would be either true *analytic apriori* (that is, it would be necessarily true because its denial either was or implied a contradiction), or *synthetic aposteriori* (that is, it would only be contingently true because of the way things worked out rather than some other way they might have worked out). For Kant, however, the judgments of both arithmetic and geometry are *apriori* all right (they're true everywhere and everywhen without fail), but their denials are not contradictions, so they are synthetic, that is, constructed by our minds. Changeux likes Kant.

Connes, however, looks to evade Kant's conclusions by contending that the issue in view is not metaphysical after all, but rather, methodological: if you countenance the existence of entities that you can't actually construct, you can generate (indirect) proofs of enormous deductive power (for example the isomorphism between Penrose tiles and the real numbers). Connes has a stronger argument, though, when noting that both formalists and intuitionists quantify over sets (that is, take sets as possible values of the bound variables of their systems). Their ontologies may not be coextensive with those of the realists, but their methods of quantification (rules of instantiation and generalization) are the same for all three foundations.

(3) Pythagoras.

Plato took much of the inspiration for his Theory of Forms from the mathematical philosophy of Pythagoras and his followers, who held that the regularities we find in nature are not so much *expressed in* numbers as they are *expressions of* numbers, or as Connes puts it, *grounded in* numbers. It is this orientation that informs (pardon the pun) Connes appeal to the "unreasonable effectiveness of mathematics." By way of example, we might think of how non-Euclidean geometries were developed by mathematicians long before Einstein concluded that our physical universe has the geometry of a Riemannian manifold. Connes cites another example: Vaughn Jones' Knot Theory, in which the find a new invariant for classifying knots: the Gordian Number, which represents the number of passes it takes to unravel a given knot. Knot theory turned out to yield an array of elegant solutions for polymer chemistry that no one could have anticipated from the theory alone.

(4) Einstein and Mathematics.

Einstein had a lot to say about the "unreasonable effectiveness of math." He once noted that "the amazing thing is that nature actually obeys our intuitions." Connes thinks Einstein matured his view about the epistemic status of mathematical judgments between 1920 and 1933. In 1920, we find Einstein contending about physics, that the "justification for a physical concept lies exclusively in its clear and unambiguous relation to facts that can be experienced." But in 1933, he writes: "The axiomatic basis of theoretical physics cannot be extracted from experience, but must be freely invented … the creative principle resides in the mathematics." Einstein's mature view was not, however, in direct conflict with his positivist proclivities in the earlier 1920 paper. Let's look a little further into his thinking, then.

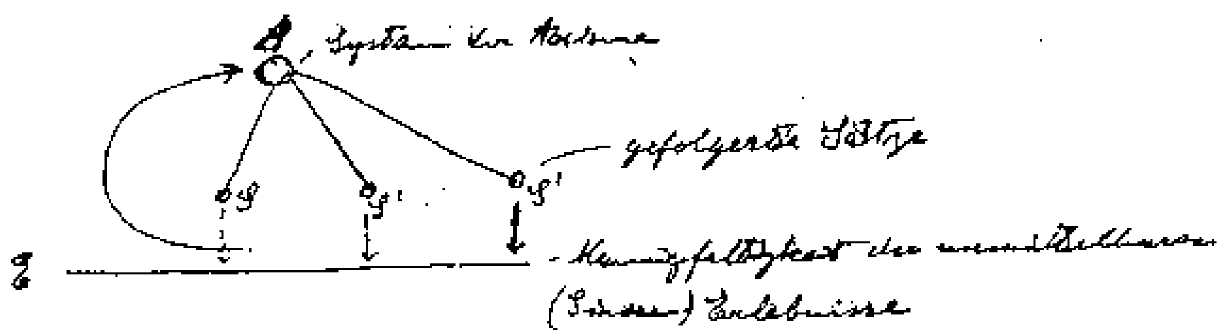
Einstein's mature epistemological views appear most clearly in his essays, "On the Method of Theoretical Physics" (1933), "Physics and Reality" (1936), and "Autobiographical Notes" (finished in 1946) . The 1946 paper is of especial historical significance because it provides several important clues as to Einstein's own sense of intellectual lineage. For example, after articulating his general view as follows,

> A proposition is correct if, within a logical system, it is deduced according to the accepted logical rules. A system has truth-content according to the certainty and completeness of its co-ordination-possibility to the totality of experience. A correct proposition borrows its "truth" from the truth-content of the system to which it belongs.

Einstein adds

> A remark to the historical development. Hume saw clearly that certain concepts, as for example that of causality, cannot be deduced from the material of experience by logical methods. Kant, thoroughly convinced of the indispensability of certain concepts, took them—just as they are selected—to be the necessary premises of every kind of thinking and differentiated them from concepts of empirical origin. I am convinced, however, that this differentiation is erroneous, i.e., that it does not do justice to the problem in a natural way. All concepts, even those which are closest to experience, are from the point of view of logic freely chosen conventions, just as in the case with the concept of causality, with which this problematic concerned itself in the first instance.

The full flavor of Einstein's neo-Kantian account of scientific reasoning is perhaps nowhere more clearly presented, however, than in a letter (written on 7 May 1952) to his longtime friend, Maruice Solovine. He says: "I view such matters schematically thus [Einstein's drawing follows]



> (1) The *E* (experiences) are given to us [represented by the horizontal line along the bottom of the figure].
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> (2) *A* are the axioms, from which we draw consequences. Psychologically the *A* rest upon the *E*. There exists, however, no logical path from the *E* to the *A*, but only an intuitive (psychological) connection which is always "subject to revocation."
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> (3) From the *A*, by a logical route, are deduced the particular assertions *S*, which deductions may lay claim to being correct.
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> (4) The *S* are referred to the *E* (test against experience). This procedure, to be exact, also belongs to the extra-logical (intuitive) sphere, because the relations between the concepts that appear in *S* and the experiences *E* are not of a logical nature.
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> These relations of the *S* to the *E*, however, are (pragmatically) much less uncertain than the relations of the *A* to the *E*. . . . If such correspondence were not obtainable with great certainty (even if not logically graspable), the logical machinery would be without any value for the comprehension of reality (example, theology).
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> The quintessence is the externally problematic connection between the world of ideas and that of experience.

In this account, we can see that, for Einstein, the logical machinery can be of value (physics) or not (theology). In this regard, Einstein was actually closer to Changeux's tool-maker than to Connes' geographer.

(5) Mathematical Models in Biology.

Changeux asserts that the biologist employs mathematics in order to accomplish two different tasks: the statistical analysis of vast sets of experimental data, on the one hand, and, on the other hand, the construction of theoretical models. Looking to neurobiology, for example, mathematics doesn't work like it does in physics, that is, by actually providing explanations of phenomena by exhaustive description. Rather, in neurobiology, models express relevant regularities; these in turn serve to help fit structures to functions, just as Mendel's laws of dominance and segregation don't predict DNA—they only imply that there are units of inheritance. In reply to this, Connes asserts that there are two stages involved in model construction: besides the descriptive stage there is a generative stage wherein mathematics adds predictive power to the descriptive elements of the model.

(6) The Auscultation of Quantum Mechanics.

Our chapter ends with a discussion of the mathematics (matrix algebra) developed by Heisenberg for modeling the quantum world. Changeux launches immediately into the familiar contention that the celebrated Indeterminacy Principle is really an epistemic worry, rather than a metaphysical discovery. The quantum physicist, according to Changeux, tends to confuse nature with h/er model of nature. Connes, however, following Neils Bohr, insists that, no, what we learn from listening to quantum mechanics is that nature is indeterminate, and that the laws of microphysics are necessarily stochastic. The reason this metaphysical conclusion is so easily misunderstood, he thinks, has nothing to do with epistemology, but rather with how we talk about phenomena: we can only talk about physical phenomena as the results of reproducible experiments, and what we reproduce in quantum mechanics are probability amplitudes of ensembles of events. In other words, experimental results cannot be considered *phenomena* unless we can reproduce them. Or, simply put: there is no science of the individual.

When Changeux replies that this looks like confusing irreproducibility with indeterminacy, Connes insists: the nature of atoms is such that they are indeterminate as regards precise position and momentum. We need our classical concepts for theory and for designing our experimental apparatus, but we can only deploy these concepts alternatively when making sense of what actually happens in experiment: we can set an apparatus up to measure either position or momentum, but never both to any arbitrary degree of precision.

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Next time, we'll look to Chapter 4: "The Neuronal Mathematician." Be well everyone, and, although I imagine you are probably quite tired by now of my continuing to say so, do remember: social distancing continues to save lives, which is presumably why we are still not in JUB 202 presently.