Some Important Derivatives

The following standard limits are needed to find the derivatives of the exponential, the sine and the cosine functions. Use your graphing calculator to find their value:

\[
\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x - 1}{x} =
\]

1. The derivative of the exponential function with base \( e \), \( y = f(x) = e^x \):

\[
\frac{de^x}{dx} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \left( \lim_{h \to 0} \frac{e^h - 1}{h} \right)
\]

The derivative of the exponential function with generic base \( a \), \( y = f(x) = a^x \)

is one application of the chain rule away if we write \( y = f(x) = a^x = e^{\ln(a^x)} = e^{x \ln a} \):

\[
\frac{da^x}{dx} = \frac{de^{x \ln a}}{dx} = \frac{de^{x \ln a}}{dx \ln a} \frac{d \ln a}{dx} =
\]

2. The derivative of the (natural) logarithm \( y = \ln x \) can now be found using the fact that \( y = \ln x \Leftrightarrow e^y = x \) and so differentiation yields:

\[
\frac{de^y}{dx} = \frac{dx}{dy} \frac{de^y}{dy} = 1 \implies e^y \frac{dy}{dx} = 1 \implies x \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{x}
\]

For generic base \( a \) we use the change of base formula \( \log_a x = \frac{\ln x}{\ln a} \) and differentiate:

\[
\frac{d \log_a x}{dx} = \frac{d}{dx} \left( \frac{\ln x}{\ln a} \right) = \frac{1}{\ln a} \frac{d}{dx} \ln x =
\]

3. The derivative of the sine function \( y = f(x) = \sin x \):

\[
f'(x) = \frac{dy}{dx} = \frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \lim_{h \to 0} \frac{(\sin x \cos h + \sin h \cos x) - \sin x}{h} =
\]

\[
= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} =
\]

Note: to distribute the sine over the sum \( x + h \) we used the sum rule for sines: \( \sin(a + b) = \sin a \cos b + \cos a \sin b \)

4. The derivative of the cosine function \( y = f(x) = \cos x \):

\[
f'(x) = \frac{dy}{dx} = \frac{d}{dx} \cos x = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} = \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} =
\]

\[
= \cos x \lim_{h \to 0} \frac{\sin h}{h} - \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} =
\]

Note: to distribute the cosine over the sum \( x + h \) we used the sum rule for cosines: \( \cos(a + b) = \cos a \cos b - \sin a \sin b \)