The Chain Rule for Derivatives.

THEOREM (Chain Rule for Derivatives). If the function \( k \) is a composition of two functions: \( k(x) = f \circ g(x) = f(g(x)) \),

then its derivative can be formed by
i. differentiating \( f \) with respect to \( g(x) \) and
ii. multiplying by \( g'(x) \):

\[
k'(x) = \left[ f[g(x)] \right]' = f'[g(x)]. g'(x)
\]

In differential notation:

\[
\frac{d(f[g(x)])}{dx} = \frac{d(f[g(x)])}{dg(x)} \cdot \frac{dg(x)}{dx}
\]

PROOF. By the definition of the derivative,

\[
k'(x) = \lim_{h \to 0} \frac{k(x + h) - k(x)}{h} = \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{h}
\]

Since \( g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \), we can write \( g(x + h) \) as \( g(x)h + g(x) \) as \( h \to 0 \) and so

\[
k'(x) = \lim_{h \to 0} \frac{f[g(x) + g'(x)h] - f[g(x)]}{h}
\]

Multiplying numerator and denominator by \( g'(x) \) and renaming \( g'(x)h = h_2 \)

\[
k'(x) = \lim_{h_2 \to 0} \frac{f[g(x) + h_2] - f[g(x)]]}{g'(x)h} \cdot g'(x)
\]

Since \( g'(x) \) is a finite number, \( g'(x)h = h_2 \to 0 \) as \( h \to 0 \),

\[
k'(x) = \lim_{h_2 \to 0} \frac{f[g(x) + h_2] - f[g(x)]}{g'(x)h} \cdot g'(x) = f'[g(x)]. g'(x)
\]

\( \square \)

EXAMPLE: \( k(x) = \sqrt{x^2 - 1} \) is a composition of the functions \( f(x) = \sqrt{x} \) and \( g(x) = x^2 - 1 \),

so its derivative is formed by ignoring the complexities of the argument of \( f \):

\[
f'(x) = \frac{1}{2\sqrt{x}} \text{ so } f'[g(x)] = \frac{1}{2\sqrt{x^2 - 1}}, \text{ then differentiating } g(x):
\]

\[
k'(x) = f'[g(x)]. g'(x) = \frac{1}{2\sqrt{x^2 - 1}}(2x) = \frac{x}{\sqrt{x^2 - 1}}.
\]

EXERCISES: Find the derivatives of the following functions with respect to the indicated variable:

1. \( f(x) = (x^2 - 2x - 3)^5 \). \( \frac{df}{dx} = \)
2. \( g(t) = \frac{3}{t^2 + 2} \). \( \frac{dg}{dt} = \)
3. \( p(s) = e^{s^3} \). \( \frac{dp}{ds} = \)
4. \( k(x) = \sqrt{\frac{1}{x^2 + 5}} \). \( \frac{dk}{dx} = \)
5. \( j(t) = \left( \frac{3}{t^2 + 2} \right)^4 \). \( \frac{dj}{dt} = \)
6. \( r(x) = 2^x \). \( \frac{dr}{dx} = \)
7. \( h(t) = \sin(\sqrt{t}) \). \( \frac{dh}{dt} = \)
8. \( q(x) = e^\sin(\sqrt{x}) \). \( \frac{dq}{dx} = \)