Systems of Linear Differential Equations

Math 3120

Differential Equations

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Consider the system of $n$ linear first order differential equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t)
\end{align*}
\]

In vector notation, this system is written as:

\[
X'(t) = A(t)X(t) + F(t)
\]

where: $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is the position vector, and $X'(t) = (x'_1(t), x'_2(t), \ldots, x'_n(t))$ is the velocity vector,

\[
A(t) = \begin{bmatrix}
a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\
a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)
\end{bmatrix}
\]

the coefficient matrix and

\[
F(t) = (f_1(t), f_2(t), \ldots, f_n(t))
\]

is the forcing vector or inhomogeneity.

A homogeneous linear system of two equations with constant coefficients,

\[
\begin{align*}
\frac{dx}{dt} &= ax(t) + by(t) \\
\frac{dy}{dt} &= cx(t) + dy(t)
\end{align*}
\]

where $a$, $b$, $c$ and $d$ are real constants.

is called a linear plane autonomous system where the term autonomous signifies that the right hand side of the system depends on $t$ only through $x$ and $y$. In vector form, the system can be written as

\[
\begin{bmatrix}
x'(t) \\
y'(t)
\end{bmatrix} =
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix},
\]

or

\[
X' = AX,
\]

where $X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

with position vector $X(t) = (x(t), y(t))$, and its derivative, the velocity vector $X'(t) = (x'(t), y'(t))$.

**Example:** The system

\[
\begin{align*}
x'(t) &= x(t) + 2y(t) \\
y'(t) &= -2x(t) + y(t)
\end{align*}
\]

with initial condition $x(0) = 0, y(0) = 2$

is written as

\[
X' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} X, X(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]
In solving a linear system, we assume both \( x(t) \) and \( y(t) \) to be exponential functions with the same constant, \( \lambda \), in the exponent:

\[
x(t) = k_1 e^{\lambda t} \quad \text{and} \quad y(t) = k_2 e^{\lambda t}
\]

or, in vector form,

\[
X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} k_1 e^{\lambda t} \\ k_2 e^{\lambda t} \end{bmatrix} = Ke^{\lambda t}
\]

The reason for choosing the common constant, \( \lambda \) in the exponent is the ensuing cancellation of the exponentials upon substitution of this exponential form into the system \( X' = AX \)

\[
\lambda Ke^{\lambda t} = AKe^{\lambda t},
\]

which gives rise to the eigenvalue/eigenvector equation: \[ AK = \lambda K \]

This, in turn, leads to the system \((A - \lambda I)K = 0\). Using the fact that system with zero right hand side \( AX=0 \) has non-trivial solutions if and only if \( \det(A) = 0 \), we obtain the characteristic equation:

\[
\det(A - \lambda I) = 0
\]

The values for the constant \( \lambda \), called the eigenvalues (or proper values) of the matrix \( A \), can be computed as the roots of the characteristic polynomial \( p(\lambda) \):

\[
\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \Rightarrow (a - \lambda)(d - \lambda) - bc = 0 , \quad \text{or} \quad p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc) = 0.
\]

The eigenvalues \( \lambda \) are the solutions of this quadratic equation and can be real and distinct, or a pair of conjugate complex numbers or, in the case of a zero discriminant, a double real eigenvalues may occur. Each distinct eigenvalue \( \lambda \) has a corresponding eigenvector \( K \) which may be found by substituting the eigenvalue into the equation \( AK = \lambda K \), and solving for the entries of \( K \):

\[
AK_1 = \lambda_1 K_1 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix} = \lambda_1 \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix} \Rightarrow (a - \lambda_1)k_{11} + bk_{12} = 0
\]

(Note that the second equation that arises: \( ck_{11} + (d - \lambda_1)k_{12} = 0 \) must lead to the same equation, since we required \( \det(A - \lambda_1 I) \) to be zero). Therefore, a possible choice for the entries of the eigenvector \( K_1 \) would be \( k_{11} = b \) and \( k_{12} = (\lambda_1 - a) \).

Thus, two solutions arise \( X_{1,2} = K_{1,2} e^{\lambda_{1,2}t} \), corresponding to distinct eigenvalues \( \lambda_{1,2} \).

**Definition:** The vectors \( X_1, X_2, \ldots, X_n \) are linearly independent if and only if

\[
c_1X_1 + c_2X_2 + \ldots + c_nX_n = 0 \quad \text{if and only if} \quad c_1 = c_2 = \ldots = c_n = 0
\]

Solutions that are not linearly independent are said to be linearly dependent.

Note that, if the solutions \( X_1, X_2, \ldots, X_n \) are linearly dependent then there exist \( c_1, c_2, \ldots, c_n \) not all zero for which \( c_1X_1 + c_2X_2 + \ldots + c_nX_n = 0 \). This means that any one solution, say \( X_2 \), can be written as a linear combination of the other solutions.
FACT: Solutions \( X_{1,2} = K_{1,2} e^{\lambda_{1,2} t} \) corresponding to distinct eigenvalues \( \lambda_{1,2} \) are linearly independent.

PROOF: Suppose the solutions \( X_1, X_2 \) are not linearly independent. Then there exist \( c_1, c_2 \) not both zero for which \( c_1 X_1 + c_2 X_2 = 0 \), or

\[
c_1 \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} k_{21} \\ k_{22} \end{bmatrix} e^{\lambda_2 t} = 0,
\]

Component wise, this means that

\[
c_1 k_{11} e^{\lambda_1 t} + c_2 k_{21} e^{\lambda_2 t} = 0.
\]

Differentiation of this equation gives the system

\[
\begin{align*}
c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} &= 0, \\
c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} &= 0,
\end{align*}
\]

which is a contradiction, since the eigenvalues are distinct. Therefore, the assumption is false and the solutions \( X_1, X_2 \) are linearly independent.

THEOREM: Let \( X_1 \) and \( X_2 \) are linearly independent solutions of the 2×2 linear system \( X'(t) = AX(t) \),

The general solution of this system is a linear combination of these solutions:

\[ X(t) = c_1 X_1 + c_2 X_2 \]

PROOF: We need to show that if \( Y(t) \) is any non-trivial solution of the system \( X'(t) = AX(t) \), then there exist unique constants \( c_1 \) and \( c_2 \) such that \( Y(t) = c_1 X_1 + c_2 X_2 \).

Equating \( Y(t) = c_1 X_1 + c_2 X_2 \),

\[ Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} k_{21} \\ k_{22} \end{bmatrix} e^{\lambda_2 t}. \]

gives

\[
\begin{bmatrix} k_{11} e^{\lambda_1 t} & k_{21} e^{\lambda_2 t} \\ k_{12} e^{\lambda_1 t} & k_{22} e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.
\]

This system has non-zero right hand side \( y_1(t) \) or \( y_2(t) \neq 0 \), since \( Y(t) \) is a non-trivial solution. Unique values for the constants \( c_1 \) and \( c_2 \) therefore exist if and only if the determinant of the matrix is non-zero:

\[
det \begin{bmatrix} k_{11} e^{\lambda_1 t} & k_{21} e^{\lambda_2 t} \\ k_{12} e^{\lambda_1 t} & k_{22} e^{\lambda_2 t} \end{bmatrix} = (k_{11} k_{22} - k_{12} k_{21}) e^{(\lambda_1 + \lambda_2) t} \\ = 0.
\]

Since an exponential is always non-zero, this leads to the requirement

\[(k_{11} k_{22} - k_{12} k_{21}) \neq 0,\]

a condition which is satisfied if \( K_1 \) and \( K_2 \) are linearly independent eigenvectors. \( \Box \)
The case of complex eigenvalues

In case the characteristic polynomial for the problem $X' = AX$ has complex conjugate roots: $\lambda = \alpha \pm \beta i$, the eigenvectors may be obtained by substitution:

$$AK_1 = \lambda_i K_1 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix} = \lambda_{i1} \begin{bmatrix} k_{11} \\ k_{12} \end{bmatrix} \Rightarrow (a - (\alpha + \beta i))k_{11} + bk_{12} = 0,$$

from which the entries of eigenvector $K_1$ may be chosen as $k_{11} = b, \quad k_{12} = (\alpha - a) + \beta i$,

so $K_1$ can be written as $K_1 = \begin{bmatrix} b \\ \alpha - a \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} = B_0 + B_1i$.

Similarly, $K_2$ can be written as $K_2 = \begin{bmatrix} b \\ \alpha - a \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \end{bmatrix} = B_0 - B_1i$.

Substituting the eigenvalues and eigenvectors into the general solution:

$$X(t) = C_1 X_1(t) + C_2 K_2 = C_1 K_1 e^{\lambda_{1}t} + C_2 K_2 e^{\lambda_{2}t}$$

we get

$$X(t) = C_1(B_0 + B_1i)e^{(\alpha + \beta i)t} + C_1(B_0 - B_1i)e^{(\alpha - \beta i)t}$$

$$= e^{\alpha t}(C_1(B_0 + B_1i)e^{\beta t} + C_1(B_0 - B_1i)e^{-\beta t})$$

Using Euler's formula: $e^{\alpha i} = \cos \theta + i \sin \theta$,

$$X(t) = e^{\alpha t}(C_1(B_0 + B_1i)(\cos \beta t + i \sin \beta t) + C_1(B_0 - B_1i)(\cos \beta t - i \sin \beta t))$$

and, collecting terms,

$$X(t) = e^{\alpha t} \left( \frac{(C_1 + C_2)(B_0 \cos \beta t - B_1 \sin \beta t) + (C_1 - C_2)i(B_1 \cos \beta t + B_0 \sin \beta t)}{C_1} \right)$$

The summary form for the general solution can be written as:

$$X(t) = C_2 X_2(t) + C_2 X_2(t)$$

where

$$X_1(t) = e^{\alpha t}(B_0 \cos \beta t - B_1 \sin \beta t) \quad \text{and} \quad X_2(t) = e^{\alpha t}(B_0 \sin \beta t + B_1 \cos \beta t)$$
The Case of the Repeat Eigenvalue.

**Theorem:** If the system $X' = AX$ has a double real eigenvalue $\lambda$ with single eigenvector $K$, a second independent solution $X_2 = (Kt + P)e^{\lambda t}$ may be obtained (in addition to the solution $X_1 = Ke^{\lambda t}$) by setting $(A - \lambda I)P = K$.

**Proof:** For $X_2 = (Kt + P)e^{\lambda t}$ to solve the system $X' = AX$, we require

$$X_2' = AX_2$$

or

$$Ke^{\lambda t} + K\lambda te^{\lambda t} + Pe^{\lambda t} = AKte^{\lambda t} + APe^{\lambda t}.$$ Collecting terms,

$$(A - \lambda I)Kte^{\lambda t} + (AP - \lambda IP - K)e^{\lambda t} = 0$$

for all values of $t$ in the domain.

Since $K$ is an eigenvector of $A$ with eigenvalue $\lambda$, we know that

$$(A - \lambda I)K = 0$$

leaving as the only condition

$$(A - \lambda I)P = K.$$ To show that the solutions $X_1$ and $X_2$ thus obtained are indeed linearly independent, we look at the determinant of the solution matrix:

$$\det[X_1|X_2] = \det[Ke^{\lambda t}|Kte^{\lambda t} + Pe^{\lambda t}] = e^{\lambda t}(k_1p_2 - k_2p_1) \neq 0,$$

since $P$ and $K$ are linearly independent.

Linear independence (LI) of the vectors $P$ and $K$ follows from the requirement $(A - \lambda I)P = K$: Suppose $P$ and $K$ are not LI, that is, one is a scalar multiple of the other. Then there exists $c$ such that $P = cK$. Then the requirement $(A - \lambda I)P = K$ leads to:

$$c(A - \lambda I)K = K$$

so

$$(A - (\lambda + 1/c) I) K = 0$$

that is, $\lambda + 1/c$ is an eigenvalue associated with the vector $K$. This contradicts the fact that $\lambda$ is the only eigenvalue for $K$. Hence $P$ and $K$ are independent, so $(k_1p_2 - k_2p_1) \neq 0$ and the solutions $X_1$ and $X_2$ are linearly independent. \qed
PROGRAM: EVAL

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PURPOSE: to compute the eigenvalues and eigenvectors of a real matrix A.

PLATFORM: Texas Instruments TI 82/83 graphing calculator

DEFINITION: An eigenvector of a matrix A is a vector which, when multiplied by A, yields a scalar multiple (the eigenvalue, \( \lambda \)) of itself: \( AK = \lambda K \)

\[
\text{Input: entries of the real constant matrix } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]

\[
\text{Output: the eigenvalues } \lambda_1 \text{ and } \lambda_2, \text{ and the corresponding eigenvectors } K_1 \text{ and } K_2
\]

PROGRAM EVAL

\[
\begin{align*}
/& \text{:Prompt A} \\
/& \text{:Prompt B} \\
/& \text{:Prompt C} \\
/& \text{:Prompt D} \\
/& \text{:STO} \\
/& \text{:Disp "REAL EVALS"} \\
/& \text{:S-T/2 L} \\
/& \text{:S+T/2 M} \\
/& \text{:Disp L,M} \\
/& \text{:Disp "EVECTS"} \\
/& \text{:Disp \{B,L-A\}} \\
/& \text{:Disp \{B,M-A\}} \\
/& \text{:If T>0} \\
/& \text{:THEN} \\
/& \text{:Disp "REAL EVALS"} \\
/& \text{:Disp S} \\
/& \text{:Disp \{S-D,C\}} \\
/& \text{:Disp \{B,S-A\}} \\
/& \text{:Else} \\
/& \text{:Disp "DOUBLE EVAL"} \\
/& \text{:Disp \{-T/2\}} \\
/& \text{:Disp \{B,S-A\}} \\
/& \text{:Disp "REAL EVECS B0"} \\
/& \text{:Disp \{B,S-A\}} \\
/& \text{:Disp "CPLX EVECS B1"} \\
/& \text{:Disp \{0, \{-T/2\}\}} \\
\end{align*}
\]

Command Locations

Prompt: pgrm ctl

If: prgm ctl; >: 2nd test

THEN: prgm ctl

Disp: prgm I/O

Else: prgm ctl