Solving second order linear DE with constant coefficients
by Variation of Parameters.

Consider the linear, second order initial value problem (IVP) consisting of the DE:
\[ y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \]  (1a)
subject to the initial conditions
\[ y(x_0) = y_0, \quad y'(x_0) = y'_0. \]  (1b)

**Theorem.** (General solution of the non-homogeneous linear DE)
The general solution of IVP (1) is
\[ y(x) = y_c(x) + y_p(x) \]  (2)
where \( y_c(x) = c_1 y_1(x) + c_2 y_2(x) \), the complementary solution, is the general solution of the homogeneous problem \( y'' + py' + qy = 0 \) and \( y_p(x) \) is any particular solution of the problem \( y'' + py' + qy = f(x) \).

**Proof.** We need to show that, if \( Y(x) \) is a solution of the DE (1a), then \( Y(x) = y_c + y_p \).

Since \( Y(x) \) is a solution of (1a), we know that \( Y'' + pY' + qY = f(x) \) (3)
Also, \( y_p \) is a solution of (1a), so that \( y_p'' + py_p' + qy_p = f(x) \) (4)
Subtracting (4) from (3) yields: \( (Y - y_p)'' + p(Y - y_p)' + q(Y - y_p) = 0 \) (5)
This means that \( Y(x) - y_p(x) \) is a solution of the homogeneous DE \( y'' + py' + qy = 0 \) (6)
and, by the representation theorem, can be written as \( y_p(x) \):
\[ Y(x) - y_p(x) = y_c(x) \] \( \rightarrow \) \( Y(x) = y_c(x) + y_p(x) \)

**Method of solving non-homogeneous second order DE.**

1. **Finding the Complementary Solution**

The complementary solution, \( y_c \), can be expressed as a linear combination of a pair of linearly independent solutions:
\[ y(x) = y_1(x) + y_2(x) \]  (7)
The independent functions \( \{y_1(x), y_2(x)\} \) are called a fundamental set of solutions.

In the case of constant coefficients, \( (p, q \) both constant), the independent solutions can be obtained from the homogeneous problem (6) by setting \( y = e^{mt} \), from which the characteristic equation \( m^2 + pm + q = 0 \) follows. This equation may have:

- distinct real roots \( m_1 \neq m_2 \) : \( y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} \) (8a)
- double real roots \( m_1 = m_2 = m \) : \( y_c = c_1 e^{mx} + c_2 xe^{mx} \) (8b)
- complex conjugate roots \( m_{1,2} = \alpha \pm \beta i \) : \( y_c = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x \) (8c)
In the case of equi-dimensional coefficients, the problem is a Cauchy-Euler equation of the form:

\[ ax^y''(x) + bxy'(x) + cy(x) = f(x), \quad a, b, c \text{ constant} \quad (9a) \]

Here, the independent solutions can be obtained by substituting \( y = x^m \) in the homogeneous DE

\[ ax^y''(x) + bxy'(x) + cy(x) = 0, \quad a, b, c \text{ constant} \quad (9b) \]

with characteristic equation: \( am(m-1) + bm + c = 0 \), which may have:

- distinct real roots \( m_1 \neq m_2 \) : \( y_c = c_1 x^{m_1} + c_2 x^{m_2} \) \quad (10a)
- double real roots \( m_1 = m_2 = m \) : \( y_c = c_1 x^m + c_2 x^m \ln x \) \quad (10b)
- complex conjugate roots \( m_{1,2} = \alpha \pm \beta i \) : \( y_c = c_1 x^\alpha \cos(\beta \ln x) + c_2 x^\alpha \sin(\beta \ln x) \) \quad (10c)

Other types of linear second order equations with variable coefficients have fundamental sets of special functions which are often found by series solutions, e.g., Bessel functions, Legendre polynomials, Airy functions, etc.

2. Finding a Particular Solution

A particular solution \( y_p \) to problem (1a): \( y'' + py' + qy = f(x) \), may be obtained by varying the parameters \( C_1 \) and \( C_2 \) in the complementary solution (7). In doing so, we set

\[ y_p = u(x)y_1 + v(x)y_2, \quad (11) \]

where \( y_1 \) and \( y_2 \) are linearly independent solutions of the homogeneous problem and \( u \) and \( v \) are unknown functions of \( x \). Using the product rule for derivatives,

\[ y_p' = uy_1' + vy_2' + u'y_1 + vy_2' + v'y_2 \quad (12) \]

Note that in this problem, there are two unknowns, the functions \( u \) and \( v \). We have imposed only one condition and are therefore at liberty to add one more constraint. To avoid a proliferation of terms in the next derivative, we set

\[ u'y_1 + v'y_2 = 0 \quad (13) \]

so the derivatives of \( y_p \) reduce to

\[ y_p' = uy_1' + vy_2' \quad \text{and} \quad y_p'' = (uy_1' + vy_2')' = u'y_1' + uy_1'' + v'y_2' + vy_2'' \quad (14) \]

Substitution of \( y_p \) and its derivatives into problem (1a) yields:

\[ \frac{(u'y_1' + uy_1'' + v'y_2' + vy_2'') + p(uy_1' + vy_2') + q(uy_1 + vy_2)}{y_p} = f(x) \quad (15a) \]

Rearranging, we get:

\[ u(y_1'' + p y_1' + q y_1) + v(y_2'' + p y_2' + q y_2) + u'y_1' + v'y_2' = f(x) \quad (15b) \]

Thus, we have a system of two equations with unknowns \( u' \) and \( v' \):

\[ \begin{cases}
  u'y_1 + v'y_2 = 0 \\
  u'y_1' + v'y_2' = f(t)
\end{cases} \Rightarrow \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix} \quad (13) \& (15b) \]

which may be solved using Cramer's Rule:

\[ u' = \frac{0 \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} - f \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \text{and} \quad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_2 & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{f y_1}{y_1 y_2' - y_2 y_1'} \quad (16) \]
A single integration yields the desired functions \( u(x) \) and \( v(x) \):

\[
\begin{align*}
u(x) &= -\int \frac{f(x)y_1(x)}{W(x)} \, dx \quad \text{and} \quad v(x) = \int \frac{f(x)y_1(x)}{W(x)} \, dx , \\
\end{align*}
\]

where the function \( W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) \) is referred to as the Wronskian of \( y_1 \) and \( y_2 \).

Constants of integration have been assigned the value zero, since any functions \( u \) and \( v \) will do.

In a computational setting, it is often advantageous to use the variable limit form of the indefinite integral: \( \int f(x)dx + C = \int f(s)ds \), which yields the following expressions for \( y_p(x) \):

\[
\begin{align*}
y_p(x) &= u(x)y_1(x) + v(x)y(x) = -y_1(x)\int_{x_0}^x \frac{f(s)y_2(s)}{W(s)} \, ds + y_2(x)\int_{x_0}^x \frac{f(s)y_1(s)}{W(s)} \, ds \\
\end{align*}
\]

**EXAMPLE:**

Solve the problem:

\[
y''(x) + \frac{1}{x}y'(x) - \frac{1}{x^2}y(x) = \sin(x)
\]

subject to the initial conditions: \( y(\frac{\pi}{2}) = 1, \quad y'(\frac{\pi}{2}) = 0 \).

Solution:

The solution of this Cauchy-Euler DE (check the fact that it's equi-dimensional), is \( y = y_c + y_p = C_1x + C_2 \frac{1}{x} - xsin x - cos x \). (Can you tell \( y_c \) from \( y_p \)?)

Applying the conditions gives: \( C_1 = \frac{1}{\pi} \), \( C_2 = \frac{3}{4} \). Verify these results.

Here's a plot of the solution:
3. The Green’s Function

Since \( y_1,2(x) \) are constants with respect to the variable of integration \( s \) in (18), we can write \( y_p(x) \) as a single integral:

\[
y_p(x) = \int_{x_0}^{x} y_1(s)y_2(x) - y_1(x)y_2(s) f(s) \, ds = \int_{x_0}^{x} G(s, x)f(s) \, ds \tag{19b}
\]

The kernel of the last integral

\[
G(s, x) = \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)}, \quad x_0 \leq s < x,
\]

is called the Green’s Function for the problem.

The Green’s function operator:

\[
H[f] = \int_{x_0}^{x} G(s, x)f(s) \, ds \tag{21}
\]

is, in some sense, an inverse of the linear operator \( L \), defined as

\[
L[y](x) = f(x) \tag{1a}
\]

since the problem

\[
L[y](x) = f(x)
\]

has particular solution

\[
y_p(x) = \int_{x_0}^{x} G(s, x)f(s) \, ds = H[f]. \tag{19b}
\]

**Example.** The problem

\[
y''(x) - y(x) = x
\]

has fundamental set \( \{y_1(x) = e^x, y_2(x) = e^{-x}\} \), so the Green’s function for this problem is

\[
G(s, x) = \frac{e^s e^{-x} - e^{-s}e^x}{-e^s e^{-x} - e^{-s}e^x} = \frac{e^{s-x} - e^{x-s}}{2}, \quad x_0 \leq s < x,
\]

which yields, for \( x_0 = 0 \), the particular solution:

\[
y_p(x) = \int_{0}^{x} G(s, x)f(s) \, ds = \frac{1}{2} \int_{0}^{x} (e^{s-x} - e^{x-s}) \, ds = \frac{1}{2} (e^x - e^{-x}) - x.
\]

So the general solution of the problem is:

\[
y(x) = y_c(x) + y_p(x) = C_1 e^x + C_2 e^{-x} + \frac{1}{2} (e^x - e^{-x}) - x = C_1 e^x + C_2 e^{-x} - x,
\]

where the exponential part of the particular solution has been absorbed in the complementary solution.