Definition: A function $f$ has a fixed point $x^*$ if $f(x^*) = x^*$

Theorem (Contraction Mapping Theorem):

Let the function $f$ be continuous on the closed interval $[a, b]$ and map the interval $[a, b]$ into itself:

$$f \in C[a, b] \quad \text{and} \quad f : [a, b] \to [a, b]$$

then $f$ has at least one fixed point $x^*$ on $[a, b]$.

Additionally, if there exists a constant $k < 1$ such that $|f'(x)| < k$ for all $x \in [a, b]$ then the fixed point $x^*$ is unique.

Furthermore, for any starting value $x_0 \in [a, b]$, the sequence

$${x_n} \to f(x_{n+1}), \quad n = 1, 2, \ldots$$

converges to the fixed point $x^*$.

Proof. To show $f$ has at least one fixed point $x^*$ on $[a, b]$, consider the case in which $f(a) = a$ or $f(b) = b$. Then $a$ or $b$ are, by definition, a fixed point and the statement is proved.

Otherwise, if $f(a) > a$, then $f(a) > a$ since $f : [a, b] \to [a, b]$, and so $f(a) - a > 0$.

For the same reason, if $f(b) < b$, then $f(b) > b$ and so $f(b) - b < 0$.

The function $h(x) = f(x) - x$ is the sum of two continuous functions, $f$ and $-x$, and thus continuous on $[a, b]$. Since $h(a) > 0$ and $h(b) < 0$, the Intermediate Value Theorem implies there exists $c \in [a, b]$ such that $h(c) = 0$, and so $f(c) = c$, a fixed point of $f$.

To show that if $|f'(x)| < k < 1$, the fixed point $x^*$ of $f$ on $[a, b]$ is unique, let $x^*$ and $x^{**}$ be distinct fixed points of $f$:

$$|x^* - x^{**}| | f(x^*) - f(x^{**})| | f'(x)| |x^* - x^{**}|$$

$$\quad \quad \quad \quad q |x^* - x^{**}| |x^* - x^{**}|$$

which is a contradiction - therefore the assumption that $x^*$ and $x^{**}$ are distinct is false.