Let me count the ways . . . .

A Problem Solving Tutorial
for General Studies Mathematics

Math 1010
Summer 2008

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1. **Counting paths in a rectangular grid.**
   Keith's school is located four blocks over and three blocks down from his home.

![Grid Diagram]

He wants to know how many different routes he can ride his bike from home to school, if no backtracking is allowed - he can only go over or down in the grid.

a. Sketch one possible route Keith can take from home to school in the above grid

b. You see that there are many different routes possible. Counting them all is no small matter! Let's first simplify the situation, and count the number of possible routes:

1 block over, and 1 block down?

_________ routes.

2 blocks over, and 1 block down?

_________ routes.

3 blocks over, and 1 block down?

_________ routes.

1 block over, and 2 blocks down?

_________ routes.

2 blocks over, and two blocks down?

_________ routes.

Three blocks over and two blocks down?

_________ routes.
c. Next, let's look for a pattern in the numbers above: transfer your results on the grid, and fill out the triangle known as Pascal’s triangle on the next page:

Pascal’s Triangle

d. Describe the pattern you see in the numbers in the triangle:

_________________________________________________________________

_________________________________________________________________

e. Fill out the triangle using this pattern.

f. If Keith’s school is located four blocks over and three blocks down from his home, how many different routes can Keith ride his bike from home to school?
   Answer: ____________.

If Keith’s school is located four blocks over and five blocks down from his home,

g. how many different routes can Keith ride his bike from home to school?
   Answer: ____________.

h. how many different routes he can ride his bike from home to school if he has to pick up his friend Mick who lives two blocks over and three blocks down from Keith’s house?
   Answer: (# ways to get to Mick’s house) _________ × (# ways to get to school from Mick’s house) _________ = ____________. 
2. The number of edges in a complete graph.

Many practical problems involve a number of locations (vertices) and connections (edges) between them. Examples are cities connected by (rail-)roads, computers connected by cables, etc. Such network constructs are commonly called graphs.

**Definition.** A graph (G) or network consists of a set V of points or vertices and a set E of connecting segments, or edges. Notation: \( G(V, E) \).

In a complete graph, every vertex is connected to all other vertices. The symbol for a complete graph with \( n \) vertices (and \( k \) edges) is \( K_n \).

**Exercises:**

a. Construct diagrams for the complete graphs \( K_n \) for \( n = 1 \ldots 7 \) in the figure below and count the number of edges.

\[
\begin{align*}
n = 1 & \quad k = ____ \\
n = 2 & \quad k = ____ \\
n = 3 & \quad k = ____ \\
n = 4 & \quad k = ____ \\
n = 5 & \quad k = ____ \\
n = 6 & \quad k = ____ \\
n = 7 & \quad k = ____
\end{align*}
\]

b. Transfer the number of edges you counted for the complete graphs \( K_n \) for \( n = 1 \ldots 7 \) in the table below and devise a pattern for this sequence.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Can you locate the numbers in the table in Pascal's triangle? If so, use the triangle to find the number of edges of \( K_{10} \).

Answer: ____________ .

d. Devise a general formula for the number of edges of \( K_n \).

Answer: ____________ .
3. **Counting amounts of money.**

In this problem, you need four distinct coins: a penny, a nickel, a dime and a quarter.

**Question:** how many different amounts can be formed using four different coins.

For instance - you can select the penny and the dime. Amount: _______ cents.

or you can select none of the four coins. Amount: _______ cents.

or you can select all of the four coins. Amount: _______ cents.

There are many more possible amounts using none, some or all of the four coins.

To answer the question, we'll start by simplifying the problem.

- Suppose you have no coins at all. How many different amounts can you form? ___

- Suppose you have just one coin. (say, the penny).
  How many different amounts can you form using no coins? ___ Using one coin? ___

- Suppose you have two coins. (say, the penny and the nickel).
  How many different amounts can you form using no coins? ___ Using one coin? ___ Using two coins? ___

- Suppose you have three coins. (say, the penny, the nickel and the dime).
  How many different amounts can you form using no coins? ___ Using one coin? ___ Using two coins? ___ Using three coins? ___

- Suppose you have four coins. (say, the penny, the nickel, the dime and the quarter).

Adding up the numbers on each line, we can make table of results:

<table>
<thead>
<tr>
<th>Number of Coins</th>
<th>Different Amounts Possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

From the pattern in the table, can you predict how many different amounts can be formed using five different coins?

**Answer:** __________.
4. **Growth Patterns**

A. A shrub exhibits the following growth pattern:
- It takes one month for a branch to mature
- Once mature, the branch produces one new branch each month

Starting with a single immature sapling, (one branch),

a. Sketch a scheme for the shrub for the first 6 months:

<table>
<thead>
<tr>
<th>month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Count the number of branches, \( f(n) \), after \( n \) months and write the result in the table below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(n) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Find a pattern in the table and describe it in words:

d. Extend the table to 12 months to find the number of branches after one year:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(n) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

d. What is the number of branches after two years?

Answer: ________________________ .

e. How many months will it take for the number of branches to exceed 1000?

Answer: ________________________ .
4B. Let’s modify the problem a bit:

A shrub exhibits the following growth pattern:
• It takes one month for a branch to mature
• Once mature, the branch produces two new branches each month.

Starting with a single immature sapling, (one branch),

a. Sketch a scheme for the shrub for the first 6 months:

<table>
<thead>
<tr>
<th>month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
</table>

b. Count the number of branches, f(n), after n months and write the result in the table below:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(n)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Find a pattern in the table and describe it in words:
_____________________________________________________________________________
_____________________________________________________________________________
_____________________________________________________________________________


g. Extend the table to 12 months to find the number of branches after one year:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(n)</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

h. What is the number of branches after two years?

Answer: ________________________ .
5. **The problem of Collatz** (or: The “3x+1 problem”)

Sometimes, a sequence or list of numbers ends up in a fixed value, regardless of your choice for a starting number.

An example of such a problem is the following sequence:

Starting with any counting number, obtain the following number by:
- dividing by 2 if the previous number is even
- multiply by 3 and add 1 if the previous number is odd.

For instance: starting (arbitrarily ) with 5, gives: \[ 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \]

Starting at 6, you get: \[ 6 \rightarrow 3 \rightarrow 10 \rightarrow 5 \rightarrow \ldots \text{ (see above, you've been there!)} \rightarrow 1 \]

Try your own number:

<table>
<thead>
<tr>
<th>example1:</th>
<th>5</th>
<th>1</th>
<th>6</th>
<th>8</th>
<th>4</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>example2:</td>
<td>6</td>
<td>3</td>
<td>10</td>
<td>5</td>
<td>16</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>your #:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>your #:</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>your #:</td>
<td></td>
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</tr>
</tbody>
</table>

(careful .... as the picture shows, some starting values (say, 27) produce huge sequences!)

The famous problem posed by the German mathematician Georg Collatz is:

**Does the above rule, for any starting number, produce a sequence that will end up in the value 1 ?**

Nobody knows ..... yet ...... This is what's called an open problem - something for mathematicians to work on!