Richardson’s Extrapolation

I approximating the value of a derivative of a function, the term extrapolation refers to the process of combining low order approximations into a result of higher order.

Commonly, the Centered Difference Quotient (CDQ) is used for the low order estimates around the node of interest, \(x_0\):

\[
\text{CDQ}(h) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}
\]  

(1)

We need to investigate the structure of the error \(e(h) = \text{CDQ}(h) - f'(x_0)\) of approximation. We start with the Taylor series of order \(n\) for \(f\), centered at \(x_0\):

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n + f^n(\xi(x))(x - x_0)^{n+1}
\]  

(2)

Evaluating this series at \(x_0 + h\):

\[
f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \cdots + \frac{f^n(x_0)}{n!}h^n + O(h^{n+1})
\]  

(3)

and at \(x_0 - h\):

\[
f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \cdots + \frac{f^n(x_0)}{n!}h^n + O(h^{n+1})
\]  

(4)

Subtracting (4) from (3) and dividing by \(2h\) yields:

\[
\frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \cdots + \frac{f^n(x_0)}{n!}h^n + O(h^{n+1})
\]  

(5)

for \(n\) even.

Rearranging, we get:

\[
f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} - \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 - \cdots - \frac{f^n(x_0)}{n!}h^n + O(h^{n+1})
\]  

(6)

which can be cast in generic form as

\[
M = N(h) + K_1h^2 + K_2h^4 + \cdots + K_jh^{2j} + O(h^{2j+1})
\]  

(7)

The extrapolation process consists of halving the mesh width

\[
M = N\left(\frac{h}{2}\right) + K_1\frac{h^2}{4} + K_2\frac{h^4}{16} + \cdots + K_j\left(\frac{h}{2}\right)^{2j} + O(h^{2j+1})
\]  

(8)

Multiplying (8) by 4 and subtracting (7) gives:

\[
M = \frac{4N\left(\frac{h}{2}\right) - N(h)}{3} + K_1h^4 + \cdots + K_jh^{2j} + O(h^{2j+1})
\]  

(9)

an approximation with error of order \(O(h^4)\).
Halving the mesh width once again,

\[ M = N_1\left(\frac{h}{2}\right) + K_1 \frac{h^4}{16} + K_3 \frac{h^6}{64} + \ldots + K_j \left(\frac{h^{2j}}{2^j}\right) + O\left(h^{2j+1}\right) \]  \hspace{1cm} (10)

we get

\[ M = \frac{16N_1\left(\frac{h}{2}\right) - N_2(h)}{15} + K_2 h^6 + \ldots + K_j h^{2j} + O\left(h^{2j+1}\right). \]  \hspace{1cm} (11)

This process can be repeated until the numerical accuracy of the results breaks down as:

\[ M = \frac{4^j N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{4^j - 1} + K_j h^{2j} + \ldots = N_j(h) + O\left(h^{2j}\right) \]  \hspace{1cm} (12)

until the numerical accuracy of the results breaks down.

Schematically, we can represent the extrapolation as:

\[
\begin{array}{cccccc}
N_0\left(\frac{h}{8}\right) & N_0\left(\frac{h}{4}\right) & N_1\left(\frac{h}{4}\right) \\
N_0\left(\frac{h}{2}\right) & N_1\left(\frac{h}{2}\right) & N_2\left(\frac{h}{2}\right) \\
N_0(h) & N_1(h) & N_2(h) & N_3(h)
\end{array}
\]

where \( N_3(h) \) is a linear combination of \( N_0(h) \), \( N_0\left(\frac{h}{2}\right) \), \( N_0\left(\frac{h}{4}\right) \) and \( N_0\left(\frac{h}{8}\right) \) and thus equivalent to a nine-points formula.