Introduction to the MAPLE Computer Algebra System.
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Part 4: The Rate of Change of a Function.

The rate at which the value of a function $f$ changes with its input over an interval $[a,b]$, is called the **Average Rate of Change (AROC)** of $f$ on $[a,b]$: 

$$ AROC[a,b] = \frac{f(b) - f(a)}{b - a} $$

The **Instantaneous Rate of Change (IROC)** of a function at a point $x = a$ results from letting the width of the interval, $h = b-a$, approach 0:

$$ IROC(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} $$

Taken at any point $x$, the $IROC(x)$ is often referred to as the **derivative** (function) of $f$.

**AROC: The Average Rate Of Change of a function**

An *average rate of change* of a function $f$ on the interval $[a,b]$:

defined as: $AROC[a,b] = (f(b)-f(a))/(b-a)$

is now straightforward to compute:

```maple
> f:=x->x^2-3*x+2;

> AROC:=(f(4)-f(2))/(4-2);
```

Graphically, the AROC is the slope of the secant line.

It is convenient to write a procedure for repeated tasks.

Here's one that graphs a function along with the secant line on an indicated interval:

```maple
> secant_line:=proc(f,a,b) local ff,m,c,p1,p2;
    ff:=unapply(f,x);
    m:= (ff(b)-ff(a))/(b-a):
    c:= ff(b)-m*b:
    plot([f,m*x+c],x=(a-1)..(b+1),color=[blue,red]):
end:
```

Let's try it for $f(x) = x^2 - 3x + 2$ on the interval $[-1,2]$:

```maple
> secant_line(x^2-3*x+2,-1,2);
```

**IROC: The Instantaneous Rate Of Change of a function**

The *instantaneous rate of change* at $x = a$ given by

$$ IROC(a) = \lim_{b \to a} \frac{(f(b) - f(a))}{(b - a)} $$

or, equivalently, with $b = a + h$

$$ IROC(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} $$

can be computed, too, as the limit of the Forward Difference Quotient $(f(a + h) - f(a)) / h$:

```maple
> FDQ:= (f(2+h)-f(2))/h;
```

which can be simplified:

```maple
> simplify(FDQ);
```

So the IROC can be found as the limit of this quantity as $h$ approaches 0:
IROC := limit(FDQ, h = 0);

Graphically, the IROC is the slope of the tangent line.
It is convenient to write a procedure for this task, too.
Here's one that graphs a function along with the tangent line on an indicated interval:

> tangent_line := proc(f, a)
local ff, h, m, c, p1, p2;
ff := unapply(f, x);
h := 0.0001:
m := (ff(a + h) - ff(a - h)) / (2 * h):
c := ff(a) - m * a:
plot([f, m * x + c], x = (a - 2) .. (a + 2), color = [blue, red]):
end:

Let's try it for \( f(x) = x^2 - 3x + 2 \) on the interval \([-1, 2]\):

> tangent_line(x^2-3*x+2,-1);

Exercises:

1. If \( f(x) \) is given as \( x/(1-x) \), compute and simplify the quantity \( (f(a + h) - f(a))/h \)
   Use the result to compute \( IROC(a) \).
   Give a graphical interpretation of the quantity \( IROC(2) \)

2. Compute the average rate of change of the function \( g(x) = x(4 - x) \) on the interval \([1, 1 + c]\)
   for \( c = 2, c = 1, c = 0.1 \) and \( c = 0.0001 \).
   What is the value of the limit of \( AROC[1, 1 + c] \) as \( c \) approaches zero?
   What is the meaning of this quantity?

3. Construct a function (procedure) that has value 1 for any input between 0 and 1 (inclusive) and has
   the value 0 anywhere else.

4. Graph the Forward Difference Quotient for the function \( f(x) = x/(x-1) \) at \( x = 2 \) as a function of \( h \)
   From this graph, what value does the function \( FDQ(h) \) take on as \( h \) approaches 0?
   What is the exact value of the limit of \( FDQ(h) \) as \( h \) approaches 0?
   What, then, is the value of the instantaneous rate of change of \( f \) at \( x = 2 \) (\( IROC(2) \))?