Introduction to Exponential Expressions

How should we read the expression $3^4$?

“Three raised to the fourth power.” We usually shorten it to “three to the fourth.”

Three to the fourth, $3^4$, is an exponential expression. An exponential expression has two parts, a base and an exponent.

The BASE is the number that we use as a multiple factor: the repeated factor.
 {Factor: a number being multiplied. Exponents mean multiplication.}

The EXPONENT counts the number of factors, telling us how many factors of the base we use in our product.

The base tells us what number to use as a factor and the exponent tells us the number of factors of that base.

In our example, $3^4$, the base is the _____ and the exponent is the _____.

We use _____ as the factor and we have _____ factors.

Since a number raised to a power is an expression, we can simplify and evaluate, but not solve.
One way to evaluate the exponential expression $3^4$ is by expanding:

<table>
<thead>
<tr>
<th>Exponent Form</th>
<th>Expanded</th>
<th>Simplified/Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3^4$</td>
<td>$3 \cdot 3 \cdot 3 \cdot 3$</td>
<td>81</td>
</tr>
</tbody>
</table>

Let’s review: In the exponential expression $5^3$:

The ____ is the repeated factor, called the BASE. The ____ is the count of factors, called the EXPONENT. The expression $5^3$, read as "five to the third power" or “five cubed,” expands to $5 \cdot 5 \cdot 5$ and evaluates to 125 [simplified or product form].

Try the following examples:

<table>
<thead>
<tr>
<th>Exponent form</th>
<th>Expanded</th>
<th>Simplified/Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5^4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3^4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$11^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
How would you write a symbolical definition to explain an exponential expression?

For any Real number \( a \) and Whole number \( n \), then

\[
\begin{align*}
\underbrace{a \cdot a \cdot \ldots \cdot a}_n
\end{align*}
\]

means \( a \cdot a \cdot \ldots \cdot a \) for \( n \) factors of \( a \).

Let’s use the expanded forms of some exponential expressions and try to develop some shortcuts (RULES) to evaluate them:

**Rule 1:** \( 2^2 \cdot 2^3 = (2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^5 \)

Shortcut: When we multiply two numbers having the same base, we _____ the exponents. \{add\}

Since the goal is to find the product, this is called the Product Rule.

Stated symbolically: For any Real number \( a \) and Whole numbers \( m \) and \( n \), \( a^m \cdot a^n = a^{m+n} \)
Rule 2: \( \frac{4^3}{4^2} = \frac{4\cdot 4\cdot 4}{4\cdot 4} = 4 \)

Shortcut: When we divide two numbers having the same base, we ______ the exponent in the denominator from the exponent in the numerator (take the exponent of the bottom base from the exponent of the top base). \{subtract\} Since the goal is to find the quotient, this is called the Quotient Rule.

Stated symbolically: For any Real number \( a, a \neq 0 \), and Whole numbers \( m > n \), \( \frac{a^m}{a^n} = a^{m-n} \).

Rule 3: \( \frac{5^3}{5^4} = \frac{5\cdot 5\cdot 5}{5\cdot 5\cdot 5\cdot 5} = \frac{1}{5} \), but using the Quotient Rule we get \( \frac{5^3}{5^4} = 5^{3-4} = 5^{-1} \). Since we know that both methods give us a correct result, we must conclude that \( 5^{-1} = \frac{1}{5} \). We now have a means of changing a base with a negative exponent to that base with a positive exponent by simply inverting the base (move the exponential expression to the
opposite side of the fraction bar and change the sign of its exponent).

Shortcut: When we need to change a negative exponent to a positive exponent, we ______ the exponential expression (take the expression from one side of the fraction bar to the other side of the fraction bar and change the sign of the exponent). {invert}

Since the goal is to change a negative exponent to a positive exponent, this is called the Negative Exponent Rule.

Stated symbolically: For any Real number \( a, a \neq 0 \), \( a^{-1} = \frac{1}{a} \).

Note that the exponent does not change, only its sign changes, so we also have \( a^{-n} = \frac{1}{a^n} \). Example: write \( a^{-3} \) without a negative exponent: \( a^{-3} = \frac{1}{a^3} \).

Rule 4: \( \frac{5^3}{5^3} = \frac{5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5} = 1 \), but using the Quotient Rule we...
get $\frac{5^3}{5^3} = 5^{3-3} = 5^0$. Since we know that both methods give us a correct result, we must conclude that $5^0 = 1$. We now have a means of finding the value of a nonzero base with a zero exponent (any nonzero number raised to the o\textsuperscript{th} power is 1).

Shortcut: When we evaluate a nonzero base with an exponent zero, we ALWAYS get _______ (for any Real # that is not zero, that number to the o\textsuperscript{th} power = 1 EVERY TIME because we are DIVIDING a number by itself). \{+1\} Since the goal is to evaluate a base with an exponent of zero, this is called the Zero Exponent Rule.

Stated symbolically: For any Real number $a$, $a \neq 0$, $a^0 = 1$. Note that this rule is based on division.

Example: evaluate $a^0$: $a^0 = 1$.

**Rule 5:** \(\left(2^2\right)^3 = \left(2 \cdot 2\right) \cdot \left(2 \cdot 2\right) \cdot \left(2 \cdot 2\right) = 2^6\)

Shortcut: When we raise an exponentiated number to a power, we ______ the exponent times the power.
Since the goal is to find the power of the number, this is called the Power Rule.
Stated symbolically: For any Real number \( a \) and Whole numbers \( m \) and \( n \), \((a^m)^n = a^{m\cdot n}\)

**Rule 6:**
\[
\left( \frac{2x^2}{5y^4} \right)^3 = \left( \frac{2x^2}{5y^4} \right) \cdot \left( \frac{2x^2}{5y^4} \right) \cdot \left( \frac{2x^2}{5y^4} \right) = \frac{8x^6}{125y^{12}}
\]

Shortcut: Rule 6 is an expansion of Rule 5 to include raising a product and a rational fraction to a power. When we raise a fraction containing an exponentiated number in the numerator, denominator, or both, to a power, we ______ each factor of the numerator, denominator, or both to the power. {raise}
Since the goal is to find the power of the fraction, this is called the Expanded Power Rule.
Stated symbolically: For any Real numbers \( a \) and \( b \), \( b \neq 0 \); variables \( x \) and \( y \), \( y \neq 0 \); and Whole numbers \( m \) and \( n \);
\[
\left( \frac{ax^m}{by^m} \right)^n = \frac{a^n x^{mn}}{b^n y^{mn}}
\]