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Public Capital Spending Shocks and the Price of Investment: Evidence from a Panel of Countries

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Abstract

A multi-sector growth model is developed where public spending affects output in one of two ways. First, the government taxes income to fund capital expenditures. Second, public capital is used in the production of both final goods and intermediate private capital goods. Inclusion of an intermediate private capital sector allows the potential of public capital investment to affect output in an indirect way that has previously not been studied in that past public investments make the accumulation process for private capital more efficient. In this case, it is shown that public investment policy is directly related to the relative price of intermediate investment goods and final goods. Using a panel of OECD countries, we find that public capital spending shocks account for a statistically important percentage of the movements in the relative price of private investment. As a result, deviations in public investment policy can account for a nontrivial portion of the cyclical variations in output even though the direct effect of public investment policy on final good production is found to be small (public capital's share in the output of final goods is only 2%).

Key words: General Equilibrium Dynamics, Investment Specific Technological Progress, Public Capital Spending Shocks, Method of Simulated Moments.

JEL category: E32, O40.

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1 Introduction

Since Aschauer (1989a; 1989b), the effects of public capital[1] investment have been intensively studied. Typically, a production function is specified and estimated where public capital directly augments the stock of private capital and labor effort. As reviewed in Gramlich (1994), Sturm, de Haan, and Kuper (1998a; 1998b), and Seitz (2001), the literature concludes that the effects of public capital on output are small. Recently, however, the empirical results of Kamps (2004) show that public capital investment significantly affects output in the reduced form, thereby suggesting that public capital may be operating to indirectly affect output. Kamps (2004) concludes that estimation results based on the direct production function approach may be obscuring public capital's total effects on output.

In this paper, we examine both theoretically and empirically the possibility that public capital indirectly affects output via the production of new private capital. Given the above scenario, increases in public capital spending can make the production of private capital more efficient, but only indirectly affects the productivity of existing capital in the production of final goods. When a factor of production positively affects productivity of the output efficiency of investment, but not the production of final goods, the technical rate of transformation between the two sectors will be a decreasing function of the factor and is called *investment specific technological progress*. As shown in Greenwood, Hercowitz, and Krusell (1997), investment specific technological progress will equal the inverse of the relative price of investment in a general equilibrium model. Therefore, a major task of our paper is to determine the contribution of public capital policy to movements in the relative price of

[1] By public capital we mean the resulting investments of a general government or nonfinancial public enterprise in one of three areas: (i) infrastructure assets, (ii) general purpose assets, (iii) and heritage assets.

private investment.

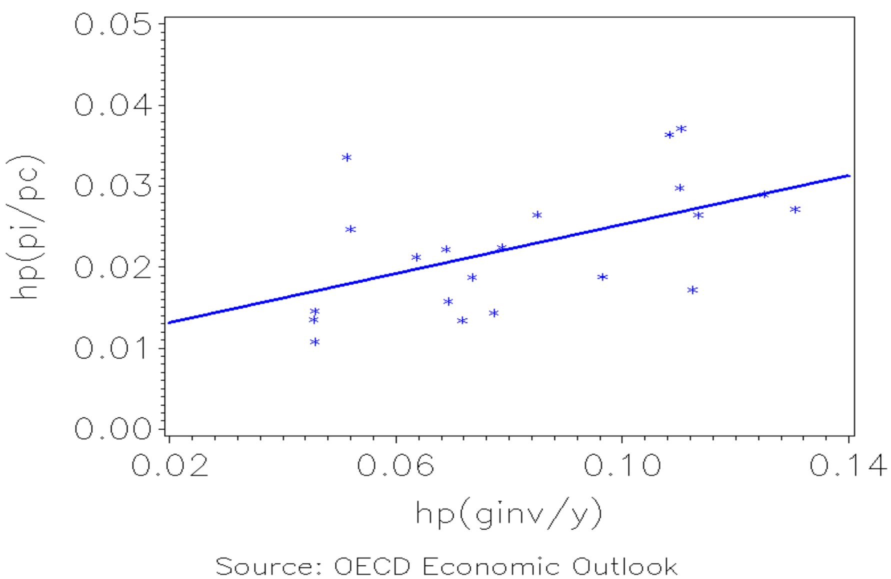
There is a preliminary evidence that the link between the distortions in the relative price of private investment and public capital expenditures may be large and important. First, the theoretical and empirical contributions of exogenous deviations in investment-specific technological change have been shown to significantly affect output over the business cycle. To our knowledge, the first paper on the subject is Greenwood, Hercowitz, and Huffman (1988). Other papers that have followed on the business cycle implications of investment-specific technological change are Fisher (1997), Christiano and Fisher (1998), Greenwood, *et. al.* (2000), and Fisher (2003). The main finding is that investment-specific technology shocks account for about 40 to 50 percent of the cyclical variations in output (Greenwood, *et. al.* 2000; Fisher, 2003). Because it is unknown whether these shocks are indeed pure technology[2], public capital investment policy has the potential to explain a large percentage of an economy's variations.

Second, it is popularly believed that the implementation of new private investment can be stimulated by previous public investments. For example, the U.S. space program (N.A.S.A.) is often credited with innovations that are subsequently embodied in private capital. Presumably, public innovations help the accumulation of private capital through "learning-by-doing." Figure 1 helpfully makes the last point by plotting the cross-country standard deviations in the HP-filtered relative price of investment against the HP-filtered public capital investment to output ratio for a mix of OECD countries. We see that public investment policy might be an important component of investment specific technology with the correlation between the two series of 53.8%.

The paper is divided into two natural parts: theoretical modeling and empirical estimation. In the theoretical modeling, public policy distorts the economy in several important ways. First, the taxes used to fund public capital investments add wedges to the costs and

[2] Fisher (2003) notes this point.

Figure 1: Scatter Plot of the Standard Deviations of the HP-filtered Relative Invesment Price and the HP-filtered Public Investment to Output Ratio. Cross Section of all OECD Countries (Correlation=53.8).



future discounted benefits for capital investments. Second, public capital is used in both the production of final consumption goods and the production of private capital as in Arrow and Kurz (1970). Finally, past public investment rates are allowed to augment the production of private investment. As discussed in Greenwood, *et. al.* (1997), this formulation (the production of private capital augmented with past investment rates) can be motivated via learning-by-doing arguments. Therefore, the purpose of the theoretical modeling and subsequent estimation is to assess the viability of the theory that previous public investment expenditures help the private sector to implement new investment projects.

Estimation of the theoretical model by the intertemporal Euler equations presents several unique problems. When technology is assumed to be embodied, stationarity of the variables (that is presumably required for estimation) would necessarily imply normalizing the aggregate variables by the level of technology which is an unobservable. Additionally, the public capital series used is estimated by the perpetual inventory method, therefore, suggesting that measurement error enters the Euler equations in a non-additive fashion. For these two reasons, General Method of Moments (GMM) will be inoperable as an estimation technique.[3]

Toward this end and as a key to the empirical analysis, we employ a version of the estimation algorithm outlined by Rotemberg and Woodford (1997) and used recently by Fuhrer (2000), Amato and Laubach (2003), and Auray and Gallès (2002). Specifically, the reduced-form processes for capital, consumption, investment price, and public policy are estimated by a structural time series model (STM) to obtain empirical estimates of the conditional distributions describing these variables. The key identifying assumption is that only shocks to investment specific technology affect the real investment price in the long run. The STM is ideal since, under the identification assumptions, it estimates the unobservable components to technology even in the presence of measurement errors and non-stationarities. The STM is then used to generate separate sets of simulated time series for the computation of the empirical distribution of impulse response functions (IRFs). Next, the theoretical relationships between the population IRFs implied by the linearized intertemporal Euler equations are replaced by the averaged simulated IRFs. In the final step of the algorithm, the simulated linearized Euler equations are summed, squared, and minimized with respect to the structural parameters of the model.

There are two important features of this algorithm to note. First, the STM result are directly used in the computation of the theoretical parameters. Therefore, the estimation procedure is relatively efficient since maximum likelihood is applied to estimate the STM. Second, because the final step of the estimation method minimizes a squared metric between simulated and sample values, the Method of Simulated Moments (MSM) of McFadden (1989) and Pakes and Pollard (1989) is being used. Operating in a MSM environment allows for the computation of the asymptotic variance-covariance matrix for the parameters and, hence, conducting hypothesis tests in the usual way.

[3] This argument was forcefully made by Sill (1992).

The results can be quickly summarized. For a panel of OECD countries, public investment shocks cannot be rejected as an important determinant for deviations in the relative price of investment. Specifically, the estimated effect of a 1% increase in public investment is found to increase the productivity of private investment by 27%. In this case, consumption and leisure are persuaded to increase while private investment rates fall. Consequently, output falls, but regains to a higher level after about three years once the more efficient private capital is installed and the effects of the taxation used by the government are mitigated. Interestingly, the direct effect of public capital on output as measured by public capitals's share in income is quantitatively small (about 2%). This implies that when policy's effect on investment productivity is exogenously set to zero that public capital spending is similar to non-productive government purchase; consumption, leisure, capital, and output are all persuaded to decrease.

The remainder of the paper is as follows. Section 2 describes the theoretical model. Section 3 presents the data and estimation methods. Section 4 presents the main results. The last section concludes.

2 The Model

The model economy is assumed to have three types of economic institutions: households, firms, and the public sector. In the model, time evolves in discrete units, called periods (which are specified to be one year long in the quantitative results reported later on).

2.1 The Households

During each period, households make decisions on consumption, supply labor, and physical capital investments. The households' problem is to maximize lifetime utility given the choice between consumption, labor hours, and loans of capital. They maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, \ell_t),$$

subject to the budget constraints and capital accumulation processes:

$$C_t + X_t/\theta_t \leq (1 - \tau_t)[R_t K_t + W_t \ell_t],$$

where β is the discount factor, C_t is consumption, ℓ_t is labor choice, K_t is capital stock, R_t is the capital rental rate, W_t is the wage rate, and τ_t is an income tax. Additionally, X_t is physical capital investment defined as: $X_t \equiv [K_{t+1} - (1 - \delta)K_t]$, and θ_t is investment specific technological progress. A key feature of this specification is that investment must occur to receive the direct benefits of technological progress.

Since there is no trend in hours worked in the data, but there is a trend in wages, we choose the momentary utility function that implies constant relative risk aversion with respect to consumption. Second, preferences for labor are assumed to be additive and separable from consumption, giving a utility function of:

$$u(C_t, \ell_t) = \frac{C_t^{1-\rho}}{1-\rho} + \omega_{1,t} \frac{(1-\ell_t)^{1-\omega_2}}{1-\omega_2}, \tag{1}$$

where the ω's give the elasticity for labor. This utility function is consistent with balanced growth only if one of the following two conditions holds: (i) $\rho = 1$ and $\omega_{1,t} = \omega_1$; or (ii) $\rho \neq 1$ and $\omega_{1,t} \equiv \omega_1 \mathcal{Z}_t^{1-\rho}$ where $\mathcal{Z}_t$ is some factor that grows at the rate of technological progress.

2.2 The Firms

The representative firm rents capital and hires labor. The firm produces consumption goods via a neoclassical constant returns to scale production function and chooses $\{K_t, \ell_t\}$ to maximize: $\pi_t = Z_t F(K_t, K_t^{pk}, N_t \ell_t) - W_t N_t \ell_t - R_t K_t$, where K_t^{pk} is public capital and Z_t is an exogenous level of neutral technological change. The firm takes as given $\{K_t^{pk}, R_t, W_t, Z_t\}$. In equilibrium, the factors of production are paid their marginal products: $Z_t F_K(t) = R_t$, $Z_t F_\ell(t) = W_t$, where $F_K(t) \equiv \partial F(K_t, K_t^{pk}, N_t \ell_t)/\partial K_t$, for example. The Cobb-Douglas form is chosen for the production technology because it is consistent with the relative constancy of income shares:

$$Y_t = Z_t F(K_t, K_t^{pk}, N_t \ell_t) = Z_t K_t^{\alpha_1} (K_t^{pk})^{\alpha_2} (N_t \ell_t)^{1-\alpha_1-\alpha_2}.$$

The capital shares are assumed to satisfy $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, and $0 < \alpha_1 + \alpha_2 < 1$. Population is assumed to be described by the following processes: $N_t = \exp(t \cdot n)$ where n is the rate of population growth.

So as to isolate the effects of investment specific technological change, the exogenous neutral technological change is assumed to follow a deterministic growth path: $Z_t = \exp(t \cdot \gamma_z)$. That is, the structure that governs neutral technological change is non-stochastic. Because changes in neutral technology shocks account for very little of the business cycle (see Fisher, 2003), the omission of a variance for neutral technological change is not likely to bias the quantitative analysis that are to follow.

2.3 The Public Sector

The public sector represents the channels through which government distorts the economy. First, income is distorted by the following tax rate:

$$\tau_t = \exp(\theta_t^g) + \exp(\theta_t^{pk}),$$

where the first part, $\exp(\theta_t^g)$, is the fraction of revenue devoted to government consumption. The second part of the total tax, $\exp(\theta_t^{pk})$, represents the fraction of revenue devoted to the accumulation of public capital. The public capital stock is taken as given by the households and firms. Expenditures on public capital must satisfy the budget constraint:

$$X_t^{pk} = \exp(\theta_t^{pk})Y_t. \tag{2}$$

where $X_t^{pk} \equiv K_{t+1}^{pk} - (1-\delta^{pk})K_t^{pk}$. Government consumption, G_t, satisfies the budget constraint:

$$G_t = \exp(\theta_t^g)Y_t.$$

Second, the value of investment specific technological change is assumed to be a function of θ_t^{pk} and is to follow the parametric form[4]:

$$\theta_t = \exp(\phi_1\bar{\theta}_{t-1}^{pk} + \theta_{t-1}^a + \theta_{t-1}^{sl})\theta_{t-1}, \tag{3}$$

where $\bar{\theta}_{t-1}^{pk} = \theta_{t-1}^{pk} - \bar{\theta}^{pk}$. The mean zero variable θ_t^a is intended to represent purely unobserved persistent shifts in changes to technology that alter the price of investment. Additionally, θ_t^{sl} is to represent stochastic slope changes in the level of technology. The effects of public capital spending on technological change are determined by the sign and magnitude of ϕ_1. If increased capital spending is associated with increases (decreases) in the productivity of private capital, then we expect $\phi_1 > 0$ ($\phi_1 < 0$); a temporarily bigger (smaller) government capital expenditure implies less (more) distortions to the price of investment.

The evolutions of remaining states are assumed to follow simple first-order autoregressive processes:

$$\begin{aligned}
\bar{\theta}_{t+1}^{pk} &= \phi_2\bar{\theta}_t^{pk} + \sigma_{pk}\varepsilon_{t+1}^{pk}, \\
\bar{\theta}_{t+1}^{g} &= \phi_3\bar{\theta}_t^{g} + \sigma_{g}\varepsilon_{t+1}^{g}, \\
\theta_{t+1}^{a} &= \phi_4\theta_t^{a} + \sigma_{a}\varepsilon_{t+1}^{a}, \\
\theta_{t+1}^{sl} &= \theta_t^{sl} + \sigma_{sl}\varepsilon_{t+1}^{sl},
\end{aligned}$$

[4] The exponential assures that the relative price of capital, that is the inverse of θ, is always positive.

where $\varepsilon_{t+1}^{i} \sim N(0,1)$. It is important to note that our level of technology is modeled as having a stochastic level and slope. The arguments for the inclusion of stochastic trends in technological advancement are presented in Fisher (2003). Essentially, the unit root assumption results from a key feature of the data that are to be presented in a following section.

2.4 A Multi-Sector Interpretation

The one-sector model studied above is a special case of a more general two-sector model. Specifically, our one-sector model where growth in capital specific technology is, in part, driven by past public investment rates is isomorphic to a two-sector model where growth is driven by externalities in the private investment goods sector. Letting sector one's resource constraint appear as:

$$C_t + G_t + X_t^{pk} = Z_t K_{1,t}^{\alpha_1} (K_{1,t}^{pk})^{\alpha_2} (N_t \ell_{1,t})^{1-\alpha_1-\alpha_2},$$

and sector two's resource constraint as:

$$X_t = \theta_t K_{2,t}^{\psi_1} (K_{2,t}^{pk})^{\psi_2} (N_t \ell_{2,t})^{1-\psi_1-\psi_2}.$$

When $\alpha_1 = \psi_1$ and $\alpha_2 = \psi_2$, the technical rate of transformation with respect to private capital between sectors one and two will equal to the inverse of θ_t. In general equilibrium, the relative prices of the two goods will always equal the technical rate of transformation. Thus, $1/\theta_t$ is the price of the intermediate investment good relative to the final consumption good.

It is apparent that when $\alpha_1 \neq \psi_1$ or $\alpha_2 \neq \psi_2$ the secular trend in the real investment price can instead be caused by other mechanisms besides the growth in investment-specific technological change. That is, as shown in Greenwood, *et. al.* (1997), the factor shares can be calibrated such that there is a secular trend in the real investment price, regardless of the existence of a trend in θ_t. However, Greenwood, *et. al.* (1997) also show that the range of values for the factor shares that give rise to price growth are implausible. The key for understanding growth and possibly deviations from the growth trend, therefore, lies in what accounts for deviations in θ_t besides the relative importance of the factors of production in the investment sector.

Inclusion of past public investment rates in θ_t is not *a priori* unreasonable. As discussed in Greenwood, *et. al.* (1997), learning-by-doing arguments can motivate the specification of past investment rates as an argument to investment specific technology. Likewise, the inclusion of past public investment rates as an argument allows for government policy to influence the private sector through learning-by-doing. Therefore, the one-sector model is equivalent to a two sector model where public investment increases the productivity of future private investment via learning-by-doing; this is the theory that is to be tested.

2.5 Characterization of the Equilibrium

The conditions for optimality for the above dynamic programming problems can be written as stochastic Euler equations:

$$-u_\ell(t) = (1-\tau_t)W_t u_c(t), \tag{4a}$$

$$\frac{u_c(t)}{\theta_t} = E_t\left\{\beta\, u_c(t+1)\left[R_{t+1}(1-\tau_{t+1}) + \frac{1}{\theta_{t+1}}(1-\delta)\right]\right\}, \tag{4b}$$

where $u_c(t) = \partial u(C_t, \ell_t)/\partial C_t$ and $u_\ell(t) = \partial u(C_t, \ell_t)/\partial \ell_t$. Market clearing, an additional assumption of the equilibrium, requires that the public capital budget constraint (2) and the aggregate resource constraint (5) hold:

$$C_t + G_t + X_t/\theta_t + X_t^{pk} = Y_t. \tag{5}$$

Since interpretation of the Euler equations will be critical to understanding the method of estimation, we take a moment to discuss the meaning of the above equations (despite their being standard conditions). The household's intratemporal first-order condition, (4a), relates the benefit of increasing labor by one unit, $W_t u_c(t)$, to the marginal cost of the lost leisure time, -$u_\ell(t)$. The intertemporal Euler equation, (4b), equates the marginal loss in utility from saving ϵ more today, $u_c(t)/\theta_t$, and the expected marginal benefit from consuming it tomorrow, where the second terms in brackets are the after-tax return on an ϵ of additional savings in physical capital.

The model estimation and solution methods require a stationary equilibrium. It is easy to show that the equilibrium conditions (2), (4a), (4b), and (5) allow for the following transformations that return stationary variables: $\mathcal{C}_t \equiv \log(C_t/\mathcal{Z}_t)$; $\mathcal{K}_t \equiv \log(K_t/(\mathcal{Z}_{t-1}\theta_{t-1}\theta_{t-1}^{sl}))$; $\mathcal{K}_t^{pk} \equiv \log(K_t^{pk}/\mathcal{Z}_{t-1})$; $\mathcal{Y}_t \equiv \log(Y_t/\mathcal{Z}_t)$; $\mathcal{G}_t \equiv \log(G_t/\mathcal{Z}_t)$; $\mathcal{W}_t \equiv \log(W_t/\mathcal{Z}_{t-1})$; $\mathcal{KY}_t \equiv \log(K_t/(Y_t\theta_{t-1}\theta_{t-1}^{sl}))$; $\mathcal{L}_t = \log(\ell_t)$; and $r_t \equiv \log(R_t\theta_{t-1}\theta_{t-1}^{sl})$,where $\mathcal{Z}_t \equiv N_t(\theta_t\theta_t^{sl})^{\alpha_1/(1-\alpha_1-\alpha_2)} \cdot (Z_t)^{1/(1-\alpha_1-\alpha_2)}$.

2.6 Solution Method

The undetermined coefficient method, described in Campbell (1994), follows a three-step procedure to produce log-linear approximations of the scaled variables $\mathcal{K}_{t+1}$, $\mathcal{K}_{t+1}^{pk}$, $\mathcal{C}_t$, and $\mathcal{L}_t$. The first step computes the first-order Taylor series expansion of the stationary versions of (2), (4a), (4b), and (5) about all of the expected values of the choice variables: $\{\bar{\mathcal{K}}, \bar{\mathcal{K}}^{pk}, \bar{\mathcal{C}}, \bar{\mathcal{L}}\}$, and the four exogenous states: $\{\bar{\theta}^g, \bar{\theta}^{pk}, \bar{\theta}^a\}$. Note that all expected values for the variables are to be replaced with their estimated evolutions from the forthcoming empirical section. The second step substitutes linear rules (guesses) for $\{\mathcal{K}_{t+1}, \mathcal{K}_{t+1}^{pk}, \mathcal{C}_t, \mathcal{L}_t\}$ into the linearized Euler Equations. The final step solves for the coefficients of the rules that set theses linearized Euler Equations to zero.

3 Data and Estimation Methods

This section describes the data that forms a panel of 21 OECD countries. The choice of variables is intended to capture the essential features of our theoretical model. Additionally, a dynamic factor model in state-space form is specified. This model is to be estimated for the reduced form co-movements of several macroeconomic variables. Finally, we detail the estimation model and methods used to obtain the structural parameters.

3.1 Data Definitions

The sample variables all derive from the *OECD Economic Outlook Database*[5] and form a 21 country panel. The countries in the panel are: Canada, Japan, Belgium, Finland, Greece, Ireland, Netherlands, Portugal, Sweden, United Kingdom, New Zealand, United States, Austria, Denmark, France, Iceland, Italy, Norway, Spain, Switzerland, and Australia. To keep the panel relatively balanced, only the years from 1960 through 2002 are used for the estimation of the models. Additionally, all expenditures are in constant prices and are in 1995 national currencies.

The two capital stock estimates used are real private net capital stock (KPV) and real public net capital stock (KGV). They are derived from the private total fixed capital formation (IPV) and government fixed capital formation (IGV) series via the perpetual inventory method described in Kamps (2004). In this method, depreciation is assumed to follow a geometric pattern that is time-varying and different across the types of capital. The remaining series identified are real GDP ($GDPV$), real private consumption expenditures (CPV), real public consumption expenditures (CGV), population (POP), gross total fixed capital formation deflator (PIT), and the private final consumption expenditure deflator (PCP).

First, we define the log of private capital/output ratio as $ky_t = \log(KPV_t/GDPV_t)$.

[5] The data set is available online at <`http://www.sourceoecd.org`>.

The log of per-capita consumption growth from period $t-1$ to t is defined as $\Delta c_t = \log(CPV_t/CPV_{t-1}) - \log(POP_t/POP_{t-1})$. Real per-capita private capital stock is defined as $k_t = \log(KPV_t/POP_t)$. Likewise, real per-capita public capital stock is defined as $k_t^{pk} = \log(KGV_t/POP_t)$. The public sector's share of investment in GDP is $\theta_t^{pk} = \log(IGV_t/GDPV_t)$. The measure of public consumption is defined as the ratio of government real consumption and real gross domestic product $\theta_t^g = \log(CGV_t/GDPV_t)$. The final variable identified from the OECD macroeconomic database is the relative price of investment that is defined by the price of investment divided by the price of consumption; the log value is denoted by $p_t = \log(PIT_t/PCP_t)$.

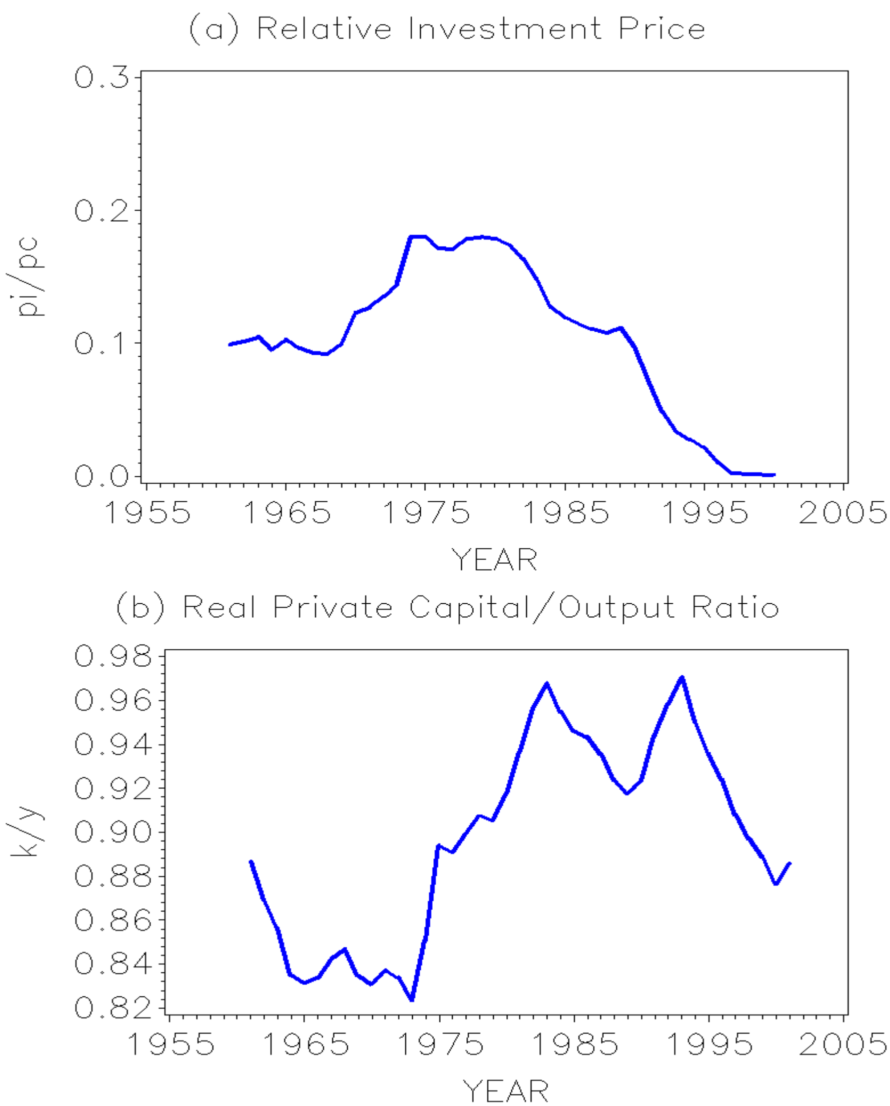
The first panel of Figure 2 displays the cross-section means for the relative price of investment which has been falling since the 1980's. At the same time, the second panel of Figure 2 shows that the private investment/output ratio increased to historically high levels. Together, the two graphs paint a picture that is empirically consistent with the facts presented in Greenwood, *et. al.* (1997) of a falling relative investment price and an increasing share of real private investment since 1980. However, prior to this period (for which the data set includes 20 years) the relative price of investment is increasing while the private capital output ratio is falling. Therefore, allowing the empirical model to include stochastically varying levels and slopes appears entirely consistent with the data.

3.2 Identification and Reduced Form Estimation

The structural time series model (STM) is used for the empirical estimate of the impact of changes in fiscal spending. The STM described in Harvey (1989) is given by the equations:

$$\begin{array}{ll} \text{(measurement equations)} & \mathbf{y}_t = \mathbf{B}\mathbf{x}_t + \mathbf{H}\boldsymbol{\xi}_t + \mathbf{w}_{t+1} \\ \text{(state equations)} & \boldsymbol{\xi}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{F}\boldsymbol{\xi}_t + \mathbf{v}_{t+1} \end{array}, \tag{6}$$

Figure 2: Means of Cross Section.



Source: OECD Economic Outlook

where $\mathbf{y}_t$ and $\mathbf{x}_t$ denote $(n \times 1)$ and $(b \times 1)$ vectors of observed variables. Alternatively, the $(r \times 1)$ vector $\boldsymbol{\xi}_t$ are latent variables. The matrices $\mathbf{A}$, $\mathbf{F}$, $\mathbf{B}$, and $\mathbf{H}$ are parameters of dimension $(r \times b)$, $(r \times r)$, $(n \times b)$, and $(n \times r)$, respectively. The vectors $\mathbf{w}_t$ and $\mathbf{v}_t$ are independent random variables defined, for all t, by $\mathbf{w}_t \sim N(\mathbf{0}, \mathbf{R})$ and $\mathbf{v}_t \sim N(\mathbf{0}, \mathbf{V})$, and $E(\mathbf{w}_t\,\mathbf{w}_{t+i}) = E(\mathbf{v}_t\,\mathbf{v}_{t+i}) = \mathbf{0}$ for all $i \neq 0$. The diagonal variance matrices $\mathbf{R}$ and $\mathbf{V}$ are of size $(n \times n)$ and $(r \times r)$, respectively.

The variables in $\mathbf{x}_t$ are a constant, time trend, and squared time trend. The variables in $\mathbf{y}$ are to be: the logged growth of per capita real consumption, the logged private capital/output ratio, logged private capital stock, logged public capital stock, logged public investment/output ratio, logged public consumption/output ratio, and the logged real price of investment. More formally, the variables are written in vector form as: $\mathbf{y}_t' = [\Delta c_t, ky_t, k_t, k_t^{pk}, \theta_t^{pk}, \theta_t^{g}, p_t]$.

The modeling of the states is intended to capture the reduced form co-movements implied by the structural model. To begin, the first factor is to represent the level of investment specific technological progress and is identified by restricting the first element of the last row of $\mathbf{H}$ to minus one ($H_{7,1} = -1$), making the state the inverse of the real price of investment. Additionally, the level of technology is modeled as a stochastic level. In this case, the first row and first column of $\mathbf{F}$ is set to one ($F_{1,1} = 1$). This econometric identification strategy for investment specific technology is essentially the same as in Fisher (2003)[6].

The second state is to represent the stationary equilibrium of private capital stock. Identification is achieved by setting the third row and second column of $\mathbf{H}$ to one ($H_{3,2} = 1$) and the third row and eighth column to (denoted $H_{3,8}$) a value which represents the theoretical factor loading of θ_{t-1} (the eighth state will be the lag of θ_t) that is given as $1+\alpha_1/(1-\alpha_1-\alpha_2)$. To control for the possibility of measurement error, the third row and third column in $\mathbf{R}$ (denoted $R_{3,3}$) is allowed to be non-zero. Private capital is also growing because of the

[6] Fisher (2003) first runs an unrestricted VAR and then imposes the restrictions to extract the impulse responses. We impose the restrictions during the estimation.

neutral technological progress. Then, the third row and second column of $\mathbf{B}$ (denoted $B_{3,2}$), the coefficient on the time trend, should be equal to neutral technological progresses factor loading: $1/(1-\alpha_1-\alpha_2)\gamma_z$. Because nonstationarity has been controlled for, private capital stock's mean is estimated by the third row and first column of $\mathbf{B}$ (denoted $B_{3,1}$).

The third state is to represent the stationary equilibrium of public capital stock. Identification is achieved by setting the fourth row and third column of $\mathbf{H}$ to one ($H_{4,3} = 1$) and the fourth row/eighth column to θ_{t-1}s factor loading $\alpha_1/(1-\alpha_1-\alpha_2)$, thereby giving the cross-equation restriction $H_{4,8} = H_{3,8} - 1$. For tractability, public capital's measurement shock is given the same variance as private capital's; $R_{4,4} = R_{3,3}$. In addition, the effect of neutral technological is the same across both types of capitals giving the restriction $B_{3,2} = B_{4,2}$. The mean of stationary public capital is estimated by the fourth row and first column of $\mathbf{B}$ (denoted $B_{4,1}$). Because private and public capital are completely described by the lagged states, the variances of their exogenous shocks in the state equations are set to zero; $V_{2,2} = V_{3,3} = 0$.

The fourth and fifth states are identified as percentages of income devoted to public investment and public consumption, respectively. They are identified by setting the fifth row and fourth column and six row and fifth column in $\mathbf{H}$ to one ($H_{5,4} = H_{6,5} = 1$). Their means are estimated in the fifth row/first column and sixth row/first column of $\mathbf{B}$ (denoted $B_{5,1}$ and $B_{6,1}$, respectively). We restrict the fourth and fifth rows of $\mathbf{F}$ to be zero, except in the case of its own past lags. In this case, $F_{4,4} = \phi_2$ and $F_{5,5} = \phi_3$ and where the corresponding elements in $\mathbf{V}$ are the variances of the shocks to the policy variables ($V_{4,4} = \sigma^2_{pk}$ and $V_{5,5} = \sigma^2_g$). Finally, the theory of a relationship between public investment and the relative price of private investment rests on it's lagged effect on the level of aggregate technology. The relevant parameter is found in the first row and fourth column of $\mathbf{F}$ ($F_{1,4} = \phi_1$).

The sixth state is to represent persistent technological change. More specifically, the sixth state is to be an autoregressive stationary process that is denoted θ^a_t; its persistence is given by the magnitude of the parameter in the sixth row and sixth column of $\mathbf{F}$ (denoted $F_{6,6} = \phi_4$) when the first row and sixth column is set to one ($F_{1,6} = 1$). The corresponding sixth row and column in $\mathbf{V}$ is the variance of the shock to θ_t^a and is denoted by $V_{6,6} = \sigma_a^2$. The seventh state is to represent stochastic slope changes to technological change. Identification is achieved by restricting both the first row and seventh column and the seventh row and seventh column of the state transition matrix to one (denoted $F_{1,7} = F_{7,7} = 1$). The corresponding seventh row and column in $\mathbf{V}$ is the variance of the shock to θ_t^{sl} and is denoted by $V_{7,7} = \sigma_{sl}^2$.

The states are represented in differenced form in the consumption growth equation. More specifically, lags are included for all the states thus allowing $\mathbf{H}$ to be augmented; this is accomplished by stacking $\mathbf{F}$ with the identity matrix $\mathbf{I}$. The coefficients on the stochastic trend and slope variables and its lagged value are restricted to $H_{1,1} = H_{1,7} = \alpha_1/(1-\alpha_1-\alpha_2)$ and $H_{1,8} = H_{1,14} = -\alpha_1/(1-\alpha_1-\alpha_2)$. As a result, the remaining consumption growth is $1/(1-\alpha_1-\alpha_2)\gamma_z$ thus giving the cross-equation restriction $B_{3,2} = B_{1,1}$. In addition to the stationary current and lagged states, the capital/output ratio includes a lagged value of the nonstationary components of investment specific technological progress state; this requires $H_{2,8} = H_{2,14} = 1$.

At this point it is important to more clearly emphasize the desired effects of identification; to extract the components of policy that will give the cyclical effects of temporary (but possibly persistent) exogenous deviations in public investment policy. However, the observed values of the policy variables are likely to have secular trends. Therefore, a time and squared time trend are included in the observation equations (denoted by the parameters $B_{5,1}$, $B_{6,1}$, $B_{5,2}$, and $B_{6,2}$). Second, policy may be responding endogenously to changes in technology. To control for the possibility of endogeneity and thus allow the extraction of the purely exogenous components, the observation equations for θ_t^{pk} and θ_t^g are augmented with the current level of θ_t^a; the effects of exogenous technology are given by the estimates of $H_{5,6}$ and $H_{6,6}$.

Estimation and inference of the STM are conducted in two steps. The first step is to compute the coefficients of the model for each country. For any country, one could apply a generalized version of least squares to (6) for the set of parameter vectors. Alternatively, a Kalman filtering approach is typically more intuitive and easier to implement. For the given parameters of the model, the filter provides the prediction error, $\boldsymbol{\eta}_{t|t-1}$, and its variance, $\mathbf{f}_{t|t-1}$. The sample log likelihood for the STM is then represented by:

$$L(\boldsymbol{\Psi}^i) = \sum_{t=1}^{T} \log((2\pi)^{-n/2}|\mathbf{f}_{t|t-1}|^{-1/2}) - \frac{1}{2}\sum_{t=1}^{T} \boldsymbol{\eta}'_{t|t-1}(\mathbf{f}_{t|t-1})^{-1}\boldsymbol{\eta}_{t|t-1},$$

and can be maximized with respect to the unknown parameters of country i (denoted by $\boldsymbol{\Psi}^i$), given initialization of the filter. Second, given the set of estimated country coefficients, an average estimate is formed by the mean group estimators of Pesaran and Smith (1995).

3.3 Structural Estimation

Estimation of the structural parameters follows the procedure outlined in Rotemberg and Woodford (1997) and used recently by Fuhrer (2000), Amato and Laubach (2003), and Auray and Gallès (2002) where the structural equations have been linearized. Then, the method relies on estimation through the conditional moments (IRFs) implied by these linearized intertemporal Euler equations.

The log-linearized intertemporal Euler equation, achieved by expansion around the model's theoretical steady state is:

$$\lambda_0 + \lambda_1\bar{\theta}_t^{pk} + \lambda_2\bar{\theta}_t^{a} = E_t\left\{\gamma_1\Delta\bar{\mathcal{C}}_{t+1} + \gamma_2\overline{\mathcal{K}\mathcal{Y}}_{t+1} + \gamma_3\bar{\theta}_{t+1}^{g} + \gamma_4\bar{\theta}_{t+1}^{pk}\right\}, \qquad (7)$$

where the bars indicate deviations from steady states and where the λs and γs are functions of the model's parameters. Additionally, after a public capital spending shock (7) holds on expectation[7] from time $t-1$, implying

$$h_t^{pk}(0,pk) = (\gamma_1/\lambda_1)h_t^c(1,pk) + (\gamma_2/\lambda_1)h_t^{ky}(1,pk) + (\gamma_3/\lambda_1)h_t^g(1,pk) + (\gamma_4/\lambda_1)h_t^{pk}(1,pk), \tag{8}$$

where the hs are the impulse response functions defined by $h_t^x(0,i) = E_t x_t - E_{t-1}x_t$ and $h_t^x(1,i) = E_t x_{t+1} - E_{t-1}x_{t+1}$ after a shock from the ith state. Additionally, $h_t^a(0,pk) = 0$ was used. One period after the shock equation (8) becomes:

$$h_t^{pk}(1,pk) = (\gamma_1/\lambda_1)h_t^c(2,pk) + (\gamma_2/\lambda_1)h_t^{ky}(2,pk) + (\gamma_3/\lambda_1)h_t^g(2,pk) + (\gamma_4/\lambda_1)h_t^{pk}(2,pk), \tag{9}$$

where $h_t^x(2,i) = E_t x_{t+2} - E_{t-1}x_{t+2}$ and $h_t^a(1,pk) = 0$. Continuing for N periods, an equation for each t can be formed as:

$$h_t^{pk}(0,pk) = \sum_{i=1}^{N}[(\hat{\gamma}_1)^i(\hat{\gamma}_4)^{i-1}h_t^c(i,pk) + (\hat{\gamma}_2)^i(\hat{\gamma}_4)^{i-1}h_t^{ky}(i,pk)] + \sum_{i=1}^{N}[(\hat{\gamma}_3)^i(\hat{\gamma}_4)^{i-1}h_t^g(i,pk)] + (\hat{\gamma}_4)^N h_t^{pk}(N,pk) \tag{10}$$

where $\hat{\gamma}_i = \gamma_i/\lambda_1$. This equation and the corresponding one for $h_t^a(0,a)$ form the basis for the estimation strategy presented in the next section.

The idea of the structural estimation method begins by substitution of estimates for the impulse responses implied by (6) into (10) since it leaves an equation in terms of the structural coefficients, ψ. To take the uncertainty of the estimates for the impulse responses into account; the responses are simulated from the distribution implied by some of the conditional moments implied by the STM. Taken together, this is the Method of Simulated Moments (MSM) of McFadden (1989) and Pakes and Pollard (1989).

[7] This discussion follows Auray and Gallès (2002).

A more detailed outline of the algorithm follows.

Definition 1 (MSM Estimation Algorthim)

- *First, set* $t = 0$.
- *Second, construct* S *separate time series of residuals of length* H *from the parameter estimates of the STM. Denote the simulated residuals* $\{\hat{\mathbf{v}}_{t+1}^{<j>}\}_{j=1}^{S}$ *where each* $\hat{\mathbf{v}}$ *is of length* H.
- *Third, from the residuals and given initial conditions on* θ_0 *construct* S *sets of length* H *synthetic time series of stationary states (i.e., holding all trends constant) and the resulting stationary observations denoted* $\{\hat{\boldsymbol{\xi}}_{t+1}^{<j>}\}_{j=1}^{S}$ *and* $\{\hat{\mathbf{y}}_{t+1}^{<j>}\}_{j=1}^{S}$*, respectively.*
- *Fourth, add a standard normal noise vector to the synthetic states and to the observations resulting in* $\{\hat{\hat{\boldsymbol{\xi}}}_{t+1}^{<j>}\}_{j=1}^{S}$ *and* $\{\hat{\hat{\mathbf{y}}}_{t+1}^{<j>}\}_{j=1}^{S}$.
- *Fifth, for each* S *use the noisy synthetic time series of observations to estimate the restricted vector autoregression (VAR) implied by the state and observation equations in (6) using the non-noisy states as instruments. Then, use the estimates to construct a set of impulse response functions:* $\{\{\hat{\mathbf{h}}_t^{<j>}(i,pk)\}_{i=1}^{N}\}_{j=1}^{S}$ *where* $\hat{\mathbf{h}}_t(i,pk) = \{\hat{h}_t^{pk}(i,pk), \hat{h}_t^{c}(i,pk), \hat{h}_t^{ky}(i,pk), \hat{h}_t^{g}(i,pk)\}$*. Also, store a subset of the reduced form estimates:* $\{\hat{\boldsymbol{\Psi}}_t^{<j>}\}_{j=1}^{S}$.
- *Finally, update* t *and return to the first step. Continue for* T *steps.*

Given the simulated impulse response functions, an MSM estimation criterion can be formed by replacing (10) with an unbiased simulator that is to be denoted:

$$\mathbf{G}_1(\psi) = \frac{1}{T \cdot S} \sum_{t=1}^{T} \sum_{j=1}^{S} g_1(\{\hat{\mathbf{h}}_t^{<j>}(i,pk)\}_{i=1}^{N}; \psi). \tag{11}$$

The moment condition is augmented by several more moments. The second set are MSM criterions for the unconditional first moments of consumption growth, the log price of investment, and the capital-output ratio; these are their theoretical steady state values implied by the model. These equations are defined as:

$$\mathbf{G}_2(\psi) = \frac{1}{T \cdot S} \sum_{t=1}^{T} \sum_{j=1}^{S} g_2(\hat{\mathbf{\Psi}}_t^{<j>}; \psi).$$

Given this setup, a consistent MSM estimator of ψ can be found by minimizing:

$$J \equiv \mathbf{G}(\psi)' \, \mathbf{W} \, \mathbf{G}(\psi),$$

where $\mathbf{G}(\psi) = [\mathbf{G}_1'(\psi), \mathbf{G}_2'(\psi)]'$ and $\mathbf{W}$ is a positive definite weighting matrix that can be optimally chosen. Because the optimal $\mathbf{W}$ depends on the unknown ψ, we use an iterative approach that first estimates with $\mathbf{W} = \mathbf{I}$. Then a weight matrix $\hat{\mathbf{W}}$ is computed from the inverse of the variance-covariance matrix of $\mathbf{G}(\psi)$ with the first round estimator $\hat{\psi}$ (Chapter 2 of Gouriéroux and Monfort, 1996). The asymptotic variance-covariance matrix for $\hat{\psi}$ is then defined on $\hat{\mathbf{W}}$ as:

$$Avar(\hat{\psi}) = T^{-1} \left((\partial \hat{\mathbf{G}} / \partial \psi)' \, \hat{\mathbf{W}} \, (\partial \hat{\mathbf{G}} / \partial \psi) \right)^{-1}.$$

For all simulations, we set $N = 50$, $S = 150$, and $T = 150$.

Also, the dimension of the parameter set is reduced by calibration. The depreciations are set to the average values used in the computation of the capital stocks, $\delta = 0.06137$ and $\delta^{pk} = 0.03192$. The growth rate of population is set at the OECD's annual rate of $n = 0.007$. The momentary utility from consumption is given where $\rho = 2$. For now, we calibrate the labor elasticity parameter at $\omega_2 = 2$ (this will be changed in a sensitivity analysis that is to follow). Finally, the calibration ω_1 is made so that the steady state labor hours are 0.33

($\omega_1 = 6.89$).

4 The Results

4.1 Reduced Form Estimation Results (STM Estimation)

The STM estimation results that are presented in Table 1 suggest several important facts about the effects of public investment. First, past public investment rates have a significant influence on the level of investment specific technological progress; the relevant parameter is estimated at $\phi_1 = 0.270$ with a significance level (p-value) close to zero. Additionally, the process that governs public investment rates indicates that it is persistent ($\phi_2 = 0.838$) and has a large percentage deviation for it's shock ($\sigma_{pk} = 0.53$) relative to the other exogenous shocks. Therefore, a 1% increase in public investment is implied to increase the productivity of private investment by about 27% and will most likely persist for several periods.

Second, investment specific technology's factor loading implies a range of values for public capital's direct share in output that is most likely small. For example, suppose labor's share in output is 50% ($1 - \alpha_1 - \alpha_2 = .50$); then public capital's share in output is about 16% (α_2 =0.17). Because 50% represents the bottom of the ranges that can be found in the literature for labor's share, it is most likely that public capital's direct share in output is small.

Finally, the upper bound on public investment policy's contribution to deviations in investment specific technology through its intermediate effect on the production of private investment can be found. To see this, suppose that the slope variable associated with the level investment specific technology is non-random ($\sigma_{sl} = 0$). Then, θ^{pk} explains 29.5%[8] of the changes in the level of investment specific technology. The remaining percentage variations are explained by technology's persistent shift variable θ^a. The reason why public

[8] With the slope's effect set to zero, investment specific technology's variance is given by: $\phi_1^2\sigma_{pk}^2/(1 - \phi_2^2) + \sigma_a^2/(1 - \phi_4^2)$.

Table 1: Selected STM Estimation Results[†].

Parameter	*Description*	*Estimate*
$H_{3,1}$	Investment specific technology's effect on private capital, $1+\alpha_1/(1-\alpha_1-\alpha_2)$	$1.67^\star$ (0.029)
$B_{3,2}$	Neutral technology's effect on private capital, $1/(1-\alpha_1-\alpha_2)\gamma_z$	$0.025^\star$ (0.009)
$B_{2,1}$	Mean of stationary private capital/output ratio, $\overline{\mathcal{KY}}$	$1.774^\star$ (0.063)
$B_{3,1}$	Mean of stationary private capital, $\overline{\mathcal{K}}$	$1.408^\star$ (0.072)
$B_{4,1}$	Mean of stationary public capital, $\overline{\mathcal{K}}^{pk}$	$0.863^\star$ (0.061)
$B_{5,1}$	Mean of public invesment's share in GDP, $\bar{\theta}^{pk}$	-2.822 (0.019)
$B_{6,1}$	Mean of government consumption's share in GDP, $\bar{\theta}^{g}$	$-1.334^\star$ (0.017)
$F_{1,4}$	Public investment's effect on the level of technology, ϕ_1	$0.270^\star$ (0.037)
$F_{4,4}$	Persistence parameter of public investment policy, ϕ_2	$0.838^\star$ (0.002)
$\sqrt{V_{4,4}}$	Standard deviation of public investment policy shock, σ_{pk}	$0.531^\star$ (0.029)
$F_{5,5}$	Persistence parameter of public consumption policy, ϕ_3	$0.845^\star$ (0.001)
$\sqrt{V_{5,5}}$	Standard deviation of public consumption policy shock, σ_g	$0.147^\star$ (0.022)
$F_{6,6}$	Persistent parameter of exogenous technological change, ϕ_4	$0.737^\star$ (0.004)
$\sqrt{V_{6,6}}$	Standard deviation of shock in persistent exogenous technological change, σ_a	$0.275^\star$ (0.039)
$\sqrt{V_{7,7}}$	Standard deviation of shock in stochastic slope component, σ_{sl}	$0.193^\star$ (0.017)
$H_{5,6}$	Technology's effect on public investment	0.033 (0.035)

[†]Estimates are mean grouped and standard error in parentheses.
*Significant at 5%. **Significant at 10%.

investment explains a large percentage despite having a factor loading of less than one is due, in part, to θ^a having a lower persistence rate ($\phi_4 = 0.737$) and a lower standard deviation for it's shock component ($\sigma_a = 0.275$). As the variance of the slope factor increases, the contribution of shocks to the public investment rate falls. Therefore, the range for public investment's effect on investment specific capital is between 0% and about 30%.

4.2 Structural Estimation Results (MSM Estimation)

The structural estimation results are presented in the first panel of Table 2 for when $\rho = 2$. The estimations show that private capital's share in output is $\alpha_1 = 0.387$ with a significance level (p-value) very close to zero. As predicted, public capital's share is estimated to be small at $\alpha_2 = 0.020$ and statistically insignificant with a p-value of 0.483. The estimates give a value for the discount factor at $\beta = 1.061$ with a significance level close to zero. The value for the discount factor appears troubling since discounting at a rate less than one is typically assumed for the existence of a solution to the dynamic programming problem. Fortunately, the notion of a discount factor in the stationary version of the economy changes to $\beta \exp(-\rho \cdot (n + \gamma_z/(1 - \alpha_1 - \alpha_2))) = 0.924$, which is less than one.

The second panel of Table 2 presents the estimation results for when $\rho = 4$. The estimations are very similar to the previous case with respect to private capital's share in output, public capital's share in output, and the growth rate of neutral technological change. The estimates give a value for the discount factor at $\beta = 1.096$ which is also similar to the case where ρ is low. Thus, the implied stationary discount factor, $\beta \exp(-\rho \cdot (n + \gamma_z/(1 - \alpha_1 - \alpha_2))) = 0.927$, is roughly the same as in the previous case.

Table 2: MSM Estimation Results†.

Parameter	*Description*	*Estimate*
Risk Aversion: $\rho = 2$		
α_1	Private capital's share in output	0.387* (0.205)
α_2	Public capital's share in output	0.020 (0.493)
β	Subjective discount factor	1.060* (0.317)
γ_z	Growth rate of neutral technological change	0.037* (0.030)
Risk Aversion: $\rho = 4$		
α_1	Private capital's share in output	0.315** (0.240)
α_2	Public capital's share in output	0.038 (0.779)
β	Subjective discount factor	1.096* (0.437)
γ_z	Growth rate of neutral technological change	0.023 (0.048)

†Standard error in parentheses.
*Significant at 5%. **Significant at 10%.

4.3 Modeling Results

Using the STM and MSM estimation parameters[9], the theoretical model is calibrated and solved for its impulse responses that are displayed in Figure 3. Consider the responses of consumption, output, and labor hours to a one-standard error shock from the public investment rate equation that are presented in panel (a) of Figure 3. Consumption responds permanently and positively to the public investment shock; this is an income effect generated by the increase in productivity caused by the shock. Output initially falls in response to the shock, but begins to slowly rises to its new long run positive value after the first period. This is also the income effect; the consumer enjoys more leisure since the accumulation of capital is more productive. In effect, households economize on their acquisition of capital

[9] Assuming $\rho = 2$.

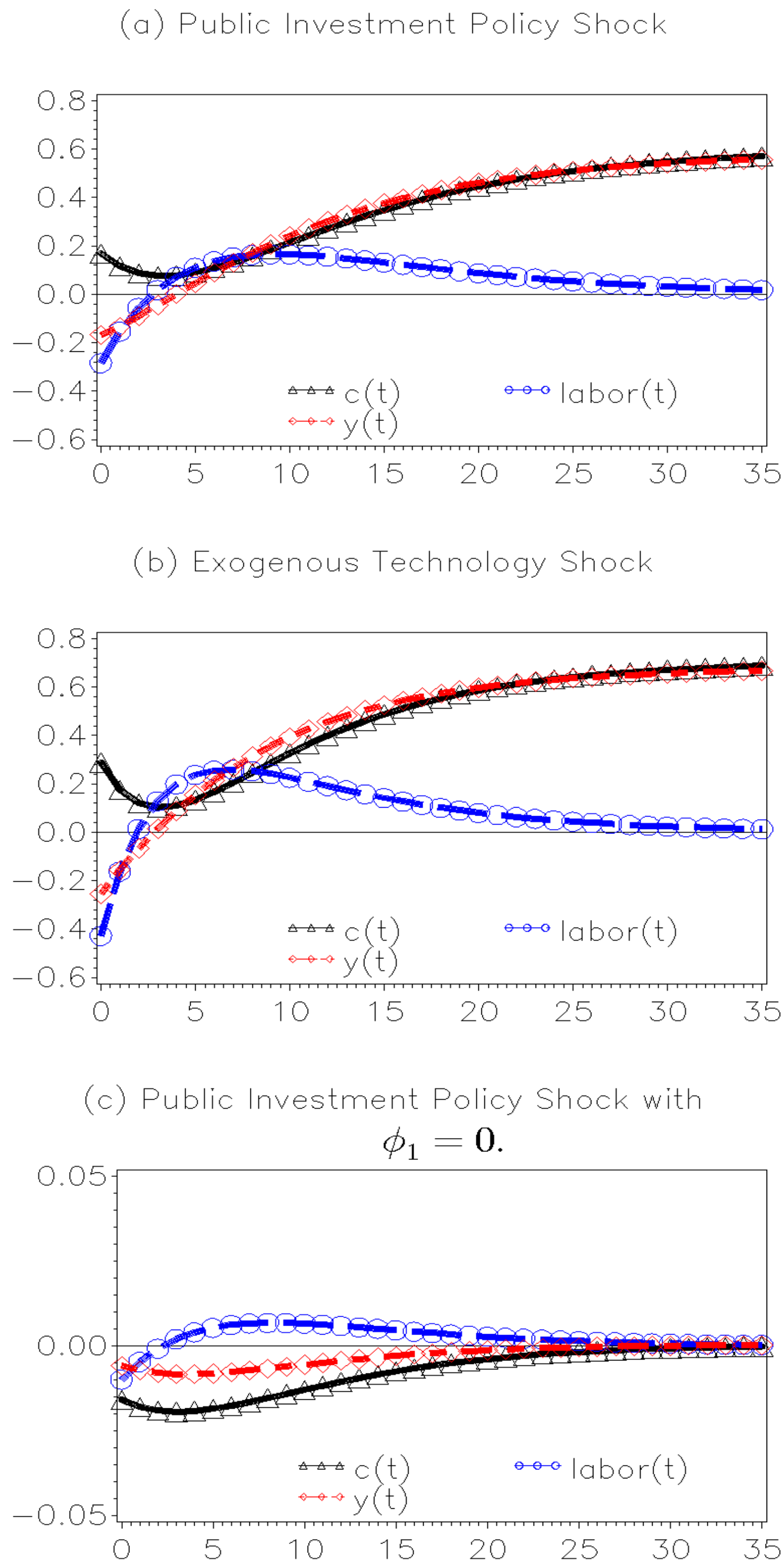
since lower rates of investment achieve higher levels of capital.

The responses of consumption, output, and labor to a one-standard error shock from exogenous investment specific technology are presented in panel (b) of Figure 3. The responses are strikingly similar to those generated from a shock to the public investment rate equation. The only difference is that the effects on consumption, output, and labor are initially more positive. Presumably, the effects of the taxation used in public investments can account for the difference.

To help us understand the differences between the two responses, consider a counterfactual experiment where past public investment rates do not affect the level of investment specific technological progress. That is, we let $\phi_1 = 0$ and display the model's responses of consumption, output, and labor effort from a one-standard error shock to the public investment rate in panel (c) of Figure 3. Notice that all three responses are initially negative. These responses correspond to a case where the benefits of public investment are dominated by the costs (*i.e.*, higher taxes) necessary to implement the public investment. Though the consumer prefers more leisure, the household is consuming and investing less due to the lower income rates. This "crowding out" effect caused by public capital investment is directly due to public capital's small share in output and is essentially the difference between the responses presented in panels (a) and (b) of Figure 3.

How much does public investment policy contribute to fluctuations in output? Consider the statistics presented in Table 3 that are the standard deviations of the HP-filtered outputs from one of three types of simulated economies (2000 simulations each). The standard deviation of output for first economy, that is denoted the baseline, is generated from the model using the structural estimates where $\rho = 2$. When public investment's effect on the level of investment specific technological progress is exogenously set to zero, $\phi_1 = 0$, the volatility of output falls from about 26% to 22%. This is roughly a 14% fall in the standard deviation of output. When the volatility of the public investment policy is set to zero, $\sigma_{pk} =$

Figure 3: The Behavior of the Theoretical Model's Aggregates.



0, output volatility remains at about 22%. Therefore, public investment primarily causes fluctuations in output through its role in the production of private investment. It is important to note that neutral technological change has been exogenously set to a deterministic time trend and thus doesn't account for any of the cyclical fluctuations in output. If neutral accounts for 50% of outputs fluctuations (as found in Greenwood, *et. al.* 2000; Fisher, 2003) then temporary deviations in public investment policy would more reasonably account for half of the 14% found by our simulations.

Table 3: Contribution of Public Investment Policy to Cyclical Output.

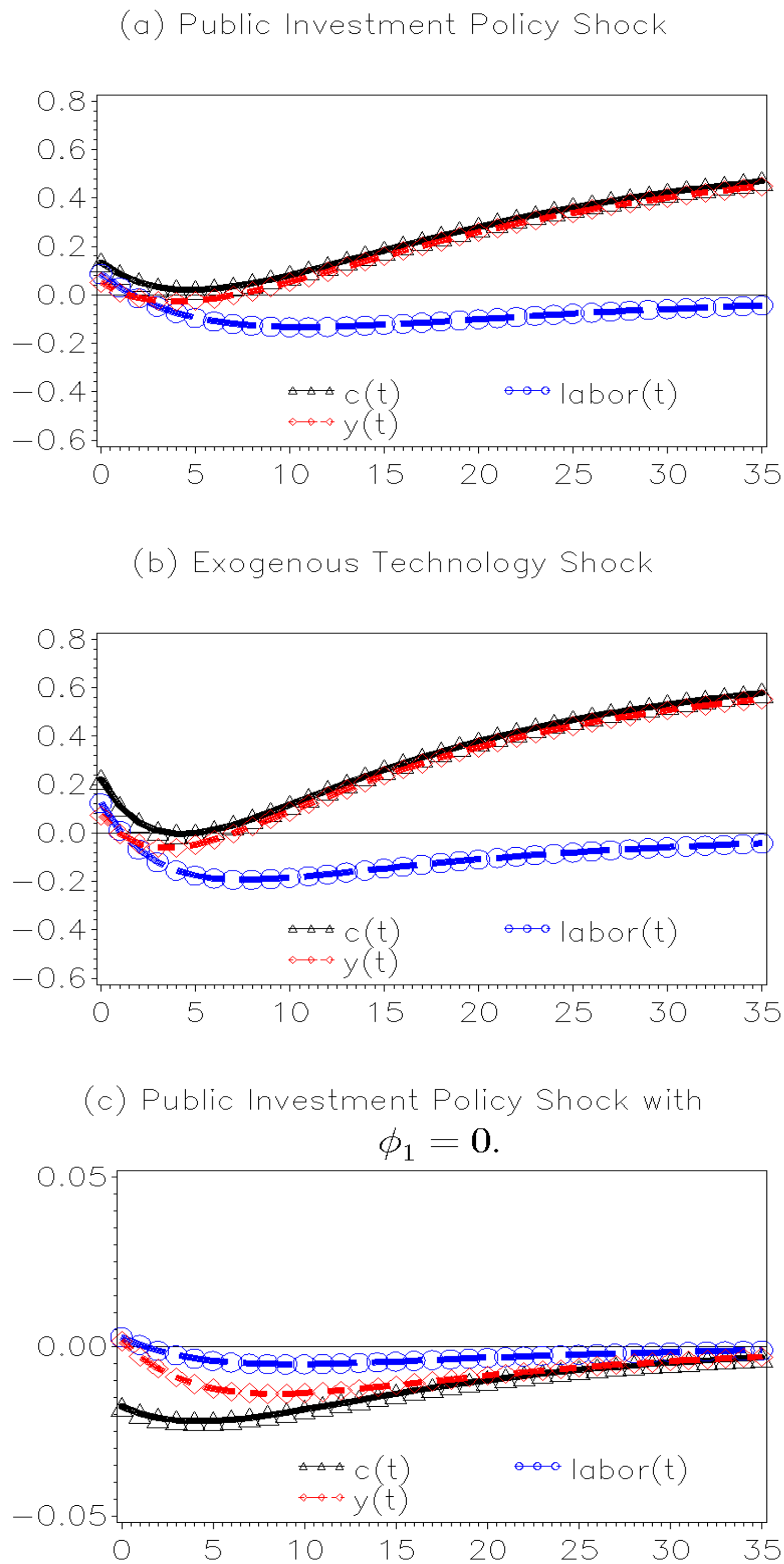
Economy	*Std. of Output*	*Relative to baseline*
baseline	0.2615	1
$\phi_1 = 0$	0.2248	0.8596
$\sigma_{pk} = 0$	0.2246	0.8588

4.4 Sensitivity Analysis

Now consider the case where the utility function is $u(C_t, \ell_t) = C_t^{1-\rho}/(1-\rho) - \omega_{1,t}(\ell_t)^{1-\omega_2}/(1-\omega_2)$. As shown in Cho and Cooley (1994), this functional form allows for a calibration where labor is substituted for leisure after a productivity shock. In fact, panel (a) of Figure 4 plots the model's impulse responses when $\rho = 2$, $\omega_1 = 0.155$, and $\omega_2 = 4$. Consumption, output, and labor hours respond positively to the shock. In labor's case, the substitution effect from the productivity shock is dominant; the consumer wants to work since the accumulation of capital is more productive. In effect, households economize on their leisure hours since the cost of leisure (the lost capital investment from not working) has increased.

The importance of this sensitivity analysis is not that a positive effect on labor hours results from a different calibration. Rather, the important result to be gleaned is in panel (c) of Figure 4. Panel (c) shows the impulse responses without past public investment rates

Figure 4: The Behavior of the Theoretical Model's Aggregates with Alternative Utility Function.



affecting the level of investment specific technological progress. Again, we see when $\phi_1 = 0$ that public investment policy is costly. Consumption falls despite the households higher labor effort. The household works more to replace the lost income from the higher tax rates needed to implement public investment. In this case, consumption and private investment are crowded-out even though output increases. In total, without public investment's intermediate effect on the production of private capital, public capital policy would be inefficient regardless of the assumptions place on the responses by household's labor/leisure efforts.

5 Conclusion

This paper addressed the macroeconomic effects of shocks to government investment policy. Investment policy shocks are modeled so that they affect the productivity of new private capital goods. Because the price of investment is equal to the technical rate of transformation between the production of new capital and the production of final goods, the policy shocks are examined in relation to fluctuations in the price of investment. To date, little is known about the actual composition of the price of investment and if it is solely a function of exogenous technology.

This research has two main conclusions. First, the direct effects of public investment policy are quantitatively small. With just one final goods sector, public capital investments crowd-out the production of consumption and private investment. Second, the indirect effects of public capital investment rates are large and positive. The estimated effect of a 1% increase in public investment is found to increase the productivity of private investment by 27%. In this case, consumption is persuaded to increase while the new more productive investment causes the private capital stock to rise.

In total, our study suggests a plausible reason for two competing empirical results which, on the one hand, assume that public capital policy affects output through the production function (Aschauer, 1989a and 1989b; Gramlich, 1994; and Sturm, *et al.*, 1998a and 1998b;

Seitz, 2001) and that public capital spending affects output indirectly (Kamps, 2004). This research shows that public investment policy affects output through its indirest and intermediary role of learning-by-doing.

Our analysis has one caveat. Both the model estimations and simulations control for mean shifts in public investment policy. The effects of permanent changes in public investment policy are, therefore, not studied. It may be the case that temporary increases in the public investment rate are correlated with permanent increases in the public investment rate. In this case, the positive effects of public investment policy may be undone. Presumably, an optimal public investment rate exists and would be a likely area for future research.

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A Appendix – Not Intended for Publication.

A.1 The Kalman Filter

The basic Kalman filter is described by the seven equations:

$$
\begin{array}{ll}
\underline{\textit{Prediction}} & \underline{\textit{Updating}} \\
\boldsymbol{\xi}_{t|t-1} = \mathbf{A}\mathbf{x}_t + \mathbf{F}\,\boldsymbol{\xi}_{t-1|t-1}, & \boldsymbol{\xi}_{t|t} = \boldsymbol{\xi}_{t|t-1} + \mathbf{K}_t\boldsymbol{\eta}_{t|t-1}, \\
\mathbf{P}_{t|t-1} = \mathbf{F}\,\mathbf{P}_{t-1|t-1}\mathbf{F}' + \mathbf{Q}, & \mathbf{P}_{t|t} = \mathbf{P}_{t|t-1} - \mathbf{K}_t\mathbf{H}\mathbf{P}_{t|t-1}, \\
\mathbf{y}_{t|t-1} = \mathbf{B}\mathbf{x}_t + \mathbf{H}\boldsymbol{\xi}_{t|t-1}, & \\
\boldsymbol{\eta}_{t|t-1} = \mathbf{y}_t - \mathbf{y}_{t|t-1}, & \\
\mathbf{f}_{t|t-1} = \mathbf{H}\mathbf{P}_{t|t-1}\mathbf{H}' + \mathbf{R}, &
\end{array}
$$

where $\boldsymbol{\xi}_{t+1|t} = E[\boldsymbol{\xi}_{t+1}|\mathbf{Y}_t]$, $\mathbf{Y}_t$ is the full information set at time t given by: $\mathbf{Y}_t \equiv (\mathbf{y}_t, \ldots, \mathbf{y}_1, \mathbf{x}_t, \ldots, \mathbf{x}_1)$, and $\mathbf{K}_t = \mathbf{P}_{t|t-1}\mathbf{H}'(\mathbf{f}_{t|t-1})^{-1}$ is the gain.

Initialization of the estimation algorithm is by definition of the matrices $\boldsymbol{\xi}_{0|0}$ and $\mathbf{P}_{0|0}$. The unconditional mean and covariance matrix of $\boldsymbol{\xi}_t$ are employed as initial values for the stationary states and are:

$$\boldsymbol{\xi}_{0|0} = (\mathbf{I} - \mathbf{F})^{-1}E(\mathbf{A}\mathbf{x}_0 + \mathbf{v}_0), \quad vec(\mathbf{P}_{0|0}) = (\mathbf{I} - \mathbf{F} \otimes \mathbf{F})^{-1}vec(\mathbf{Q}).$$

When $\boldsymbol{\xi}_t$ is non-stationary, we can treat $\boldsymbol{\xi}_{0|0}$ as a parameter to be estimated. Because $\boldsymbol{\xi}_{0|0}$ is no longer a random variable, $\mathbf{P}_{0|0}$ should be set to set equal to 0 in the non-stationary rows and columns (Harvey, 1989). Once the parameters are estimated, the algorithm may be re-initialized by setting $\boldsymbol{\xi}_{0|0} = \hat{\boldsymbol{\xi}}_{0|0}$ and $\mathbf{P}_{0|0} = cov(\hat{\boldsymbol{\xi}}_{0|0})$.

A.2 Stationary Eulers

Rearranging the intertemporal condition (4b) and assuming functional form (1) for utility gives a stationary intertemporal Euler Equation:

$$1 = E_t\left\{\beta e^{-\rho\Delta\mathcal{C}_{t+1}}\frac{e^{-\rho\cdot(n+s_2\gamma_z)}}{e^{\rho\cdot s_1(\phi_1\bar{\theta}^{pk}_t+\theta^a_t)}}\left[\begin{array}{c}(1-e^{\theta^g_{t+1}}-e^{\theta^{pk}_{t+1}})\alpha_1 e^{(\alpha_1-1)\mathcal{K}_{t+1}}e^{\alpha_2\mathcal{K}^{pk}_{t+1}}\times \\ (e^n e^{\mathcal{L}_{t+1}})^{1-\alpha_1-\alpha_2}+(1-\delta)\frac{1}{e^{\phi_1\bar{\theta}^{pk}_t+\theta^a_t}}\end{array}\right]\right\}, \tag{12}$$

where $\Delta\mathcal{C}_{t+1} \equiv \mathcal{C}_{t+1} - \mathcal{C}_t$, $s_1 \equiv \alpha_1/(1-\alpha_1-\alpha_2)$, and $s_2 \equiv 1/(1-\alpha_1-\alpha_2)$. The intratemporal Euler Equation (4a) in stationary form is:

$$\omega_1(1-e^{\mathcal{L}_t})^{-\omega_2} = \left\{e^{-\rho\mathcal{C}_t}\frac{e^{-(n+s_2\gamma_z)}}{e^{s_1(\phi_1\bar{\theta}^{pk}_{t-1}+\theta^a_{t-1})}}\left[\begin{array}{c}(1-e^{\theta^g_t}-e^{\theta^{pk}_t})\times \\ (1-\alpha_1-\alpha_2)e^{\alpha_1\mathcal{K}_t}e^{\alpha_2\mathcal{K}^{pk}_t}(e^n e^{\mathcal{L}_t})^{-\alpha_1-\alpha_2}e^n\end{array}\right]\right\}, \tag{13}$$

where $\omega_1 \equiv \omega_{1,t}/\mathcal{Z}_t^{1-\rho}$ is some parameter that is to be calibrated. The stationary version of the resource constraint (5) is given by:

$$e^{\mathcal{C}_t} + e^{\mathcal{G}_t} + \left\{ \begin{array}{c} e^{\mathcal{K}_{t+1}} - (1-\delta)e^{\mathcal{K}_t}\frac{e^{-(n+s_2\gamma_z)}}{e^{(1+s_1)(\phi_1\bar{\theta}^{pk}_{t-1}+\theta^a_{t-1})}} + \\ e^{\mathcal{K}^{pk}_{t+1}} - (1-\delta^{pk})e^{\mathcal{K}^{pk}_t}\frac{e^{-(n+s_2\gamma_z)}}{e^{s_1(\phi_1\bar{\theta}^{pk}_{t-1}+\theta^a_{t-1})}} \end{array} \right\} = \left\{ \begin{array}{c} \frac{e^{-(n+s_2\gamma_z)}}{e^{s_1(\phi_1\bar{\theta}^{pk}_{t-1}+\theta^a_{t-1})}} \times \\ e^{\alpha_1\mathcal{K}_t}e^{\alpha_2\mathcal{K}^{pk}_t}(e^n e^{\mathcal{L}_t})^{1-\alpha_1-\alpha_2} \end{array} \right\}. \tag{14}$$

The public capital investment constraint (2) in stationary form is given by:

$$e^{\mathcal{K}^{pk}_{t+1}} - (1-\delta^{pk})e^{\mathcal{K}^{pk}_t}\frac{e^{-(n+s_2\gamma_z)}}{e^{s_1(\phi_1\bar{\theta}^{pk}_{t-1}+\theta^a_{t-1})}} = \left\{ \begin{array}{c} e^{\theta^{pk}_t}\frac{e^{-(n+s_2\gamma_z)}}{e^{s_1(\phi_1\bar{\theta}^{pk}_{t-1}+\theta^a_{t-1})}} \times \\ e^{\alpha_1\mathcal{K}_t}e^{\alpha_2\mathcal{K}^{pk}_t}(e^n e^{\mathcal{L}_t})^{1-\alpha_1-\alpha_2} \end{array} \right\} \tag{15}$$

Alternatively, the intertemporal condition (4b) can be written as a function of the capital-output ratio as:

$$0 = E_t\left\{\beta e^{-\rho\Delta\mathcal{C}_{t+1}}\frac{e^{-\rho\cdot(n+s_2\gamma_z)}}{e^{\rho\cdot s_1(\phi_1\bar{\theta}^{pk}_t+\theta^a_t)}}\left[\begin{array}{c}(1-e^{\theta^g_{t+1}}-e^{\theta^{pk}_{t+1}})\alpha_1 e^{-\mathcal{KY}_{t+1}} + \\ (1-\delta)\frac{1}{e^{\phi_1\bar{\theta}^{pk}_t+\theta^a_t}}\end{array}\right]\right\} - 1$$

Linearization around the steady states gives an equation of the form:

$$\lambda_0 + \lambda_1\bar{\theta}^{pk}_t + \lambda_2\bar{\theta}^a_t = E_t\left\{\gamma_1\Delta\bar{\mathcal{C}}_{t+1} + \gamma_2\overline{\mathcal{KY}}_{t+1} + \gamma_3\bar{\theta}^g_{t+1} + \gamma_4\bar{\theta}^{pk}_{t+1}\right\}$$

where the λ's and γ's are defined as:

$$\begin{aligned}
\lambda_0 &= -\beta e^{-\rho\cdot(n+s_2\gamma_z)}\left[(1-e^{\bar{\theta}^g}-e^{\bar{\theta}^{pk}})\alpha_1 e^{-\overline{\mathcal{KY}}} + (1-\delta)\right] + 1 \\
\lambda_1 &= \rho s_1\phi_1\beta e^{-\rho\cdot(n+s_2\gamma_z)}\left[(1-e^{\bar{\theta}^g}-e^{\bar{\theta}^{pk}})\alpha_1 e^{-\overline{\mathcal{KY}}} + (1-\delta)\right] + (1-\delta)\phi_1\beta e^{-\rho\cdot(n+s_2\gamma_z)} \\
\lambda_2 &= \rho s_1\beta e^{-\rho\cdot(n+s_2\gamma_z)}\left[(1-e^{\bar{\theta}^g}-e^{\bar{\theta}^{pk}})\alpha_1 e^{-\overline{\mathcal{KY}}} + (1-\delta)\right] + (1-\delta)\beta e^{-\rho\cdot(n+s_2\gamma_z)} \\
\gamma_1 &= -\rho\beta e^{-\rho\cdot(n+s_2\gamma_z)}\left[(1-e^{\bar{\theta}^g}-e^{\bar{\theta}^{pk}})\alpha_1 e^{-\overline{\mathcal{KY}}} + (1-\delta)\right] \\
\gamma_2 &= -\alpha_1\beta e^{-\rho\cdot(n+s_2\gamma_z)}\left[(1-e^{\bar{\theta}^g}-e^{\bar{\theta}^{pk}})\alpha_1 e^{-\overline{\mathcal{KY}}}\right] \\
\gamma_3 &= -\beta e^{-\rho\cdot(n+s_2\gamma_z)}\left[e^{\bar{\theta}^g}\alpha_1 e^{-\overline{\mathcal{KY}}}\right] \\
\gamma_4 &= -\beta e^{-\rho\cdot(n+s_2\gamma_z)}\left[e^{\bar{\theta}^{pk}}\alpha_1 e^{-\overline{\mathcal{KY}}}\right]
\end{aligned}$$

A.3 State-Space Form

Given $\mathbf{y}_t' = [\Delta c_t, ky_t, k_t, k_t^{pk}, \theta_t^{pk}, \theta_t^{g}, p_t]$ and $\mathbf{x}_t' = [1, (t-\bar{t}), (t-\bar{t})^2]$, the state-space form is given by the following sequence of matrices:

$$\mathbf{H}_1 = \begin{bmatrix} H_{1,1} & H_{1,2} & H_{1,3} & H_{1,4} & H_{1,5} & H_{1,6} & H_{1,7} \\ 0 & H_{2,2} & H_{2,3} & H_{2,4} & H_{2,5} & H_{2,6} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & H_{5,6} & 0 \\ 0 & 0 & 0 & 0 & 1 & H_{6,6} & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_2 = \begin{bmatrix} -H_{1,1} & -H_{1,2} & -H_{1,3} & H_{1,11} - H_{1,4} & -H_{1,5} & H_{1,13} - H_{1,6} & -H_{1,7} \\ 1 & H_{2,9} & H_{2,10} & H_{2,11} & H_{2,12} & H_{2,13} & 1 \\ H_{3,8} & 0 & 0 & 0 & 0 & 0 & H_{3,14} \\ H_{4,8} & 0 & 0 & 0 & 0 & 0 & H_{4,14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_3 = \begin{bmatrix} 0 & 0 & 0 & -H_{1,11} & 0 & -H_{1,13} & 0 \\ 0 & 0 & 0 & H_{2,18} & 0 & H_{2,20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 \end{bmatrix}$ and

$$\mathbf{B} = \begin{bmatrix} B_{1,1} & 0 & 0 \\ B_{2,1} & B_{2,2} & 0 \\ B_{3,1} & B_{3,2} & 0 \\ B_{4,1} & B_{4,2} & 0 \\ B_{5,1} & B_{5,2} & B_{5,3} \\ B_{6,1} & B_{6,2} & B_{6,3} \\ 0 & 0 & 0 \end{bmatrix}$$

Additionally

$$\mathbf{F}_1 = \begin{bmatrix} 1 & 0 & 0 & F_{1,4} & 0 & 1 & 1 \\ 0 & F_{2,2} & F_{2,3} & F_{2,4} & F_{2,5} & F_{2,6} & 0 \\ 0 & F_{3,2} & F_{3,3} & F_{3,4} & F_{3,5} & F_{3,6} & 0 \\ 0 & 0 & 0 & F_{4,4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{5,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{6,6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{F}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{2,11} & 0 & F_{3,13} & 0 \\ 0 & 0 & 0 & F_{3,11} & 0 & F_{3,13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}$ and

$$\mathbf{V}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{4,4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{5,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{6,6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{7,7} \end{bmatrix}$$

where $\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.