An explicit local basis for C^1 cubic spline spaces over a triangulated quadrangulation

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Abstract

Let $S_3^1(\diamondsuit)$ be the bivariate C^1-cubic spline space over a triangulated quadrangulation $\diamondsuit$. In this paper, an explicit representation of a locally supported basis of $S_3^1(\diamondsuit)$ is given using the interpolation conditions at vertices.

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1. Introduction

For a connected polygonal domain Ω in $\mathbb{R}^2$, let $\triangle$ be a triangulation of Ω, and by this, we mean that the complement of $\triangle$ relative to Ω consists of a finite number of triangles such that none of the vertices of any triangle lies on the interior of any edge of other triangles. For integers d and r with $0 \leqslant r \leqslant d - 1$, we define $S_d^r(\triangle)$ to be the vector space of C^r functions which are polynomials with total degree at most d restricted to each triangle of $\triangle$. The space $S_d^r(\triangle)$ is called a bivariate spline space with degree d and smoothness order r.

Practical applications of bivariate spline spaces include function approximation, surface fitting, computer aided geometric design (CAGD), and numerical solution of partial differential equations. One of the most basic problems in the study of spline spaces is to determine their dimensions and to find their bases.

Dimension problem was initiated with a conjecture by Strang [38,39]. The first result was given by Morgan and Scott [29] on a dimension formula and an explicit basis for bivariate spline space $S_d^1(\triangle)$

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with $d \geqslant 5$. Later, Schumaker [35] gave a lower bound formula for the dimension of the spaces $S_d^r(\triangle)$. Alfeld and Schumaker [5] proved that Schumaker's lower bound is in fact the dimension of $S_d^r(\triangle)$ for $d \geqslant 4r + 1$. Together with Piper, they constructed an explicit basis for the space $S_d^r(\triangle)$, $d \geqslant 4r + 1$ in [3]. By carefully working with the smoothness conditions in terms of B-net representation of spline functions, Hong [17] proved that Schumaker's lower bound formula indeed gives the dimension for the space $S_d^r(\triangle)$ when $d \geqslant 3r + 2$. By a clever application of the B-net approach, Alfeld, Piper and Schumaker [4] also extended the Morgan–Scott results to the space $S_4^1(\triangle)$.

Careful readers can see that almost all of the results in the dimension problem mentioned above merely come from those cases when degree d is relatively high versus smoothness order r. In practical application, due to the simplicity and efficiency in calculation, the spline spaces with lower degrees versus smoothness orders are more important and favorable. For example, spaces $S_d^r(\triangle)$ with $d = 2, 3, 4$ for $r = 1$ and with $d = 5, 6, 7$ for $r = 2$. However, the dimensions of the spaces $S_d^2(\Delta)$, $d = 2, 3, 4$ have not yet been determined. For $r = 1$, as mentioned above, the dimension of $S_4^1(\triangle)$ has been successfully established by Alfeld et al. [4] and a local basis for optimal approximation purpose has been constructed by Chui and Hong [8] using a so-called edge swapping method. However, the dimensions of $S_2^1(\triangle)$ and $S_3^1\triangle)$ are still open, although several results are obtained for some special triangulations by Ye [41], Liu [25,26], Zhang and Lin [42], and Liu and Hong [28] for examples. As pointed out by Morgan–Scott [30] via a counter example, the dimension of $S_2^1(\triangle)$ not only depends on the topological invariants of $\triangle$ such as the total number of the vertices, triangles and edges, but also heavily depends on the geometric property of triangulation $\triangle$. More counter examples and further studies can be found in [6,11,12,37]. As to the space $S_3^1(\triangle)$, up to now, no one knows whether its dimension depends on the geometric structure of the triangulation or not. Alfeld [2] sought the possibility to determine the dimension of $S_3^1(\triangle)$ by using techniques developed for the solution of Four Color Map problem.

Due to the extreme difficulty in the study of $S_d^r(\Delta)$ for small values of d, some special triangulations are attracted attentions. The very applicable triangulations are various kinds of refinements of original triangulation. The idea of refinement was originally introduced by Clough and Tocher [10] where a triangle in the original triangulation is subdivided into three subtriangles at any interior point of the triangle. Following their work, Ciarlet [9] and Percell [31] showed that they can uniquely construct a C^1 piecewise cubic polynomial function on $\triangle_{CT}$ to interpolate the function values and the gradient values at the vertices of $\triangle$ and the normal derivatives at all the edges of $\triangle$, where $\triangle_{CT}$ denotes Clough–Tocher's refinement. They actually obtained that $\dim(S_3^1(\triangle_{CT})) = 3|V| + |E|$.

Then, Powell and Sabin [32] proposed a kind of refinement where each triangle in the original triangulation is subdivided into 6 subtriangles (normally, the interior point is chosen as the incenter of the triangle), and they found that there exists a unique C^1 quadratic spline function on $\triangle_{PS1}$ which interpolates the given function values and the gradient values at the vertices of $\triangle$. This means that $\dim(S_2^1(\triangle_{PS1})) = 3|V|$. In the same paper, Powell and Sabin also studied another refinement $\triangle_{PS2}$ where each triangle in $\triangle$ is divided into 12 subtriangles and they proved that there exists a unique C^1 quadratic spline function on $\triangle_{PS2}$ which interpolates the given function values and the gradient values at the vertices of $\triangle$ and the normal derivatives at all the edges of $\triangle$. This in fact means $\dim(S_2^1(\triangle_{PS2})) = 3|V| + |E|$.

Recently, because of the important application in scattered data interpolation in CAGD and also the application of multiresolution approximation, the refinement technique becomes more and more

attractive. On the basis of Clough–Tocher's and Powell–Sabin's refined triangulation, many kinds of refinements are proposed. Heindl [16] constructed a piecewise quadratic C^1 interpolating spline in $\triangle_{\text{PS2}}$. By using the three medians of three edges and the barycentre of each triangle in $\triangle_{\text{PS2}}$, an explicit basis of $S_2^1(\triangle_{\text{PS2}})$ with support containing at most one vertex of the original triangulation $\triangle$ in its interior has been constructed by Chui and He [7]. Alfeld [1] used Clough–Tocher's refinement twice to subdivide each triangle of $\triangle$ into 9 subtriangles, and then constructed an interpolating spline function in $S_5^2(\triangle_A)$. Sablonniere [33] used Powell–Sabin's refinement method to subdivide each triangle of $\triangle$ into six subtriangles at the center of its inscribed circle and constructed interpolating spline in $S_5^2(\triangle_{\text{PS1}})$. Wang [40] subdivided each triangle of $\triangle$ into 7 subtriangles in a very special way and constructed an interpolating spline in $S_5^2(\triangle_W)$. Gao [15] also employed Clough–Tocher's method and constructed an interpolating spline in $S_6^2(\triangle_{\text{CT}})$. Lai [21] generalized Sablonniere's $S_5^2(\triangle_{\text{PS1}})$ to $\hat{S}_5^2(\triangle_{\text{PS1}})$ so that C^3 data is not required for interpolation.

Besides various kinds of refinements of triangulation, a triangulated refinement of quadrangulation is also studied. Let $\Diamond$ be a quadrangulation of Ω which consists of nondegenerate convex quadrilaterals. By adding the two diagonals of each quadrilateral, a special kind of triangulation, called triangulated quadrangulation and denoted by $\Diamond\!\!\!\!+$, can be obtained. In 1964–1965, two Belgium engineers Sanders [34] and Fraejis de Veubek [14] originally constructed a C^1 cubic spline finite element on $\Diamond\!\!\!\!+$ and applied it in structure analysis. However, during the next thirty years, not many people followed these two pioneers' work. Recently, Lai [22] reconstructed the space $S_3^1(\Diamond\!\!\!\!+)$ by extending the well-known four-directional mesh (or Type-II triangulation) $\triangle_{mn}^{(2)}$ into the triangulated quadrangulation $\Diamond\!\!\!\!+$ of a general polygonal domain. In his paper, an interpolation scheme was given and the optimal approximation order of $S_3^1(\Diamond\!\!\!\!+)$ was also demonstrated. This is an extension of the results on the space $S_3^1(\triangle_{mn}^{(2)})$ studied in [20].

As pointed out by Lai and Schumaker [23], due to the optimal approximation order, smaller dimension, and less number of triangles in $\Diamond\!\!\!\!+$, $S_3^1(\Diamond\!\!\!\!+)$ is better than the typical splitting-triangulation based spline spaces, such as $S_3^1(\triangle_{\text{CT}})$, $S_2^1(\triangle_{\text{PS}})$, and super spline space $S_5^{1,2}(\triangle)$ [36] in practical applications. Lai and Wenston [24] have used it to solve steady state Navier–Stokes equations. Hong and Schumaker [19] applied elements of $S_3^1(\Diamond\!\!\!\!+)$ into surface compression very recently.

However, we need to point out that part of the B-net ordinates of the local supported basis given by Lai [22] is defined by a set of interpolation conditions implicitly, this makes the application of the basis inconvenient and inefficient. In this paper, it is shown that, by using some new relationship given by Hong and Liu [18,27], all the Bézier net ordinates can be analytically solved from those interpolation conditions, therefore the locally supported basis can be explicitly given. This can make the interpolation and approximation schemes using the splines of $S_3^1(\Diamond\!\!\!\!+)$ more efficient.

2. Smoothness conditions of bivariate splines

Let $T_1 = [v_0, v_1, v_2]$ be a planar non-degenerate triangle with vertices $v_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 0, 1, 2$. Let $\xi = (\xi_0, \xi_1, \xi_2)$ be the barycentric coordinate of $\mathbf{x} \in \mathbb{R}^2$ with respect to the triangle T_1. Set

$$\lambda_i := y_j - y_k, \quad \mu_i := -(x_j - x_k), \quad v_i := x_j y_k - y_j x_k,$$

with (i, j, k) being a cycling of the subscripts in cyclic order $0 \to 1 \to 2 \to 0$. Clearly, the following identities hold:

$$\lambda_0 + \lambda_1 + \lambda_2 = 0, \quad \mu_0 + \mu_1 + \mu_2 = 0, \quad \nu_0 + \nu_1 + \nu_2 = 0.$$

Then the barycentric coordinate can be expressed as

$$\begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix} = \frac{1}{A^{(1)}} \begin{bmatrix} \nu_0 & \lambda_0 & \mu_0 \\ \nu_1 & \lambda_1 & \mu_1 \\ \nu_2 & \lambda_2 & \mu_2 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix},$$

where

$$A^{(1)} = 2\text{area}[v_0, v_1, v_2] = \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}$$

is twice of the oriented area of the triangle T_1.

Let $p(x, y)$ be a bivariate polynomial with total degree d on triangle T_1. According to the theory of B-net (see [13] for example), $p(x, y)$ can be written as

$$p(x, y) = p(\xi_0, \xi_1, \xi_2) = \sum_{\alpha_0 + \alpha_1 + \alpha_2 = d} b^{(1)}_{\alpha_0 \alpha_1 \alpha_2} \frac{d!}{\alpha_0! \alpha_1! \alpha_2!} \xi_0^{\alpha_0} \xi_1^{\alpha_1} \xi_2^{\alpha_2},$$

where $b^{(1)}_{\alpha_0 \alpha_1 \alpha_2}$ is called the B-net ordinate of p with respect to T_1.

We have the following (see [18,27]):

$$\begin{bmatrix} \dfrac{\partial p}{\partial x} \\ \dfrac{\partial p}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \xi_0}{\partial x} & \dfrac{\partial \xi_1}{\partial x} \\ \dfrac{\partial \xi_0}{\partial y} & \dfrac{\partial \xi_1}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial p}{\partial \xi_0} \\ \dfrac{\partial p}{\partial \xi_1} \end{bmatrix}$$

$$= \frac{1}{A^{(1)}} \begin{bmatrix} \lambda_0 & \lambda_1 \\ \mu_0 & \mu_1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial p}{\partial \xi_0} \\ \dfrac{\partial p}{\partial \xi_1} \end{bmatrix}$$

$$= \sum_{|\alpha| = d - 1} \frac{d}{A^{(1)}} \binom{|\alpha|}{\alpha} \begin{bmatrix} \lambda_0 & \lambda_1 \\ \mu_0 & \mu_1 \end{bmatrix} \begin{bmatrix} \triangle_{13} b^{(1)}_{\alpha} \\ \triangle_{23} b^{(1)}_{\alpha} \end{bmatrix} \xi_0^{\alpha_0} \xi_1^{\alpha_1} \xi_2^{\alpha_2}, \tag{1}$$

where

$$\triangle_{13} b^{(1)}_{\alpha} := b^{(1)}_{\alpha_0 + 1 \alpha_1 \alpha_2} - b^{(1)}_{\alpha_0 \alpha_1 \alpha_2 + 1},$$

$$\triangle_{23} b^{(1)}_{\alpha} := b^{(1)}_{\alpha_0 \alpha_1 + 1 \alpha_2} - b^{(1)}_{\alpha_0 \alpha_1 \alpha_2 + 1},$$

and

$$
\begin{bmatrix} \dfrac{\partial^2 p}{\partial x^2} \\[2mm] \dfrac{\partial^2 p}{\partial y^2} \\[2mm] \dfrac{\partial^2 p}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial \xi_0}{\partial x}\right)^2 & \left(\dfrac{\partial \xi_1}{\partial x}\right)^2 & 2\dfrac{\partial \xi_0}{\partial x}\dfrac{\partial \xi_1}{\partial x} \\[2mm] \left(\dfrac{\partial \xi_0}{\partial y}\right)^2 & \left(\dfrac{\partial \xi_1}{\partial y}\right)^2 & 2\dfrac{\partial \xi_0}{\partial y}\dfrac{\partial \xi_1}{\partial y} \\[2mm] \dfrac{\partial \xi_0}{\partial x}\dfrac{\partial \xi_0}{\partial y} & \dfrac{\partial \xi_1}{\partial x}\dfrac{\partial \xi_1}{\partial y} & \dfrac{\partial \xi_0}{\partial x}\dfrac{\partial \xi_1}{\partial y} + \dfrac{\partial \xi_1}{\partial x}\dfrac{\partial \xi_0}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 p}{\partial \xi_0^2} \\[2mm] \dfrac{\partial^2 p}{\partial \xi_1^2} \\[2mm] \dfrac{\partial^2 p}{\partial \xi_0 \partial \xi_1} \end{bmatrix}
$$

$$
= \frac{1}{(A^{(1)})^2} \begin{bmatrix} \lambda_0^2 & \lambda_1^2 & 2\lambda_0\lambda_1 \\ \mu_0^2 & \mu_1^2 & 2\mu_0\mu_1 \\ \lambda_0\mu_0 & \lambda_1\mu_1 & \lambda_0\mu_1 + \lambda_1\mu_0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 p}{\partial \xi_0^2} \\[2mm] \dfrac{\partial^2 p}{\partial \xi_1^2} \\[2mm] \dfrac{\partial^2 p}{\partial \xi_0 \partial \xi_1} \end{bmatrix}
$$

$$
= \sum_{|\alpha|=d-2} \frac{d(d-1)}{(A^{(1)})^2} \binom{|\alpha|}{\alpha} \begin{bmatrix} \lambda_0^2 & \lambda_1^2 & 2\lambda_0\lambda_1 \\ \mu_0^2 & \mu_1^2 & 2\mu_0\mu_1 \\ \lambda_0\mu_0 & \lambda_1\mu_1 & \lambda_0\mu_1 + \lambda_1\mu_0 \end{bmatrix}
$$

$$
\times \begin{bmatrix} \triangle_{13} \triangle_{13} \, b_\alpha^{(1)} \\ \triangle_{23} \triangle_{23} \, b_\alpha^{(1)} \\ \triangle_{13} \triangle_{23} \, b_\alpha^{(1)} \end{bmatrix} \xi_0^{\alpha_0} \xi_1^{\alpha_1} \xi_2^{\alpha_2}. \tag{2}
$$

Applying formula (1) and (2) to a bivariate cubic polynomial $p(x, y)$ on T_1, we have

$$
p|_{v_0} = b_{300}^{(1)}, \tag{3}
$$

$$
D_x p|_{v_0} = \frac{3}{A^{(1)}}[\lambda_0 b_{300}^{(1)} + \lambda_1 b_{210}^{(1)} + \lambda_2 b_{201}^{(1)}], \tag{4}
$$

$$
D_y p|_{v_0} = \frac{3}{A^{(1)}}[\mu_0 b_{300}^{(1)} + \mu_1 b_{210}^{(1)} + \mu_2 b_{201}^{(1)}], \tag{5}
$$

$$
D_{xx} p|_{v_0} = \frac{6}{(A^{(1)})^2}[\lambda_0^2 b_{300}^{(1)} + \lambda_1^2 b_{120}^{(1)} + \lambda_2^2 b_{102}^{(1)} + 2\lambda_0\lambda_2 b_{201}^{(1)} + 2\lambda_1\lambda_0 b_{210}^{(1)} + 2\lambda_2\lambda_1 b_{111}^{(1)}], \tag{6}
$$

$$
D_{yy} p|_{v_0} = \frac{6}{(A^{(1)})^2}[\mu_0^2 b_{300}^{(1)} + \mu_1^2 b_{120}^{(1)} + \mu_2^2 b_{102}^{(1)} + 2\mu_0\mu_2 b_{201}^{(1)} + 2\mu_1\mu_0 b_{210}^{(1)} + 2\mu_2\mu_1 b_{111}^{(1)}], \tag{7}
$$

$$D_{xy}p|_{v_0} = \frac{6}{(A^{(1)})^2}[\lambda_0\mu_0 b_{300}^{(1)} + \lambda_1\mu_1 b_{120}^{(1)} + \lambda_2\mu_2 b_{102}^{(1)}$$

$$+ (\lambda_0\mu_2 + \lambda_2\mu_0)b_{201}^{(1)} + (\lambda_1\mu_0 + \lambda_0\mu_1)b_{210}^{(1)} + (\lambda_2\mu_1 + \lambda_1\mu_2)b_{111}^{(1)}]. \tag{8}$$

For convenience, we recall some notation used in [4]. Associated with the triangulation $\triangle$, let

$$P \equiv P_d := \bigcup_{l=1}^{N}\{P_{ijk}^{[l]} = (iv_0^{[l]} + jv_1^{[l]} + kv_2^{[l]})/d, \ i+j+k=d\},$$

where $v_0^{[l]}$, $v_1^{[l]}$, $v_2^{[l]}$ are the vertices of the lth triangle $[v_0^{[l]}, v_1^{[l]}, v_2^{[l]}]$ in counterclockwise order. The points in P are called the domain points. We call the point $P_{ijk}^{[l]}$ is of distance $(d-i)$ from vertex $v_0^{[l]}$. Similarly, we call the point $P_{ijk}^{[l]}$ is of distance i from the edge opposite $v_0^{[l]}$. The ring of order m around the vertex v is

$$R_m(v) = \{\text{points which are distance } m \text{ from } v\},$$

and the disk of order m around v is

$$D_m(v) = \bigcup_{j=0}^{m} R_j(v).$$

3. The explicit local basis for $S_3^1(\lozenge\!\!\!\!\!\!/)$

Let $\lozenge$ be a convex quadrangulation of Ω. Subdivide each quadrilateral of $\lozenge$ into four triangles by adding two diagonals. This results in a special triangulation, called a triangulated quadrangulation of Ω. The corresponding C^1-cubic spline space $S_3^1(\lozenge\!\!\!\!\!\!/)$ is called the bivariate C^1-cubic spline space over $\lozenge\!\!\!\!\!\!/$.

Let $V_\lozenge = \{v\}$ be the collection of all vertices of $\lozenge$ and $E_\lozenge = \{e\}$ be the collection of all edges of $\lozenge$. The locally supported basis of $S_3^1(\lozenge\!\!\!\!\!\!/)$ constructed in [22] includes three vertex splines $V_{v,s}$, $s \in \{(0,0),(1,0),(0,1)\}$ at each vertex $v \in V$, and one edge spline V_e corresponding to each edge $e \in E$. All of these $3|V_\lozenge| + |E_\lozenge|$ spline functions form a local basis of the space $S_3^1(\lozenge\!\!\!\!\!\!/)$. The vertex splines $V_{v,s}$ with $s = (s_1,s_2) \in \{(0,0),(1,0),(0,1)\}$ are defined by the following interpolation conditions

$$D^t V_{v,s}(u) = \delta_{v,u}\delta_{t,s}, \quad t \in \{(0,0),(1,0),(0,1)\}, \ u \in V_\lozenge, \tag{9}$$

$$D_e^{(1,1)} V_{v,s}(v_{e,1}) = 0, \quad e \in E_\lozenge, \tag{10}$$

and the edge spline V_e with $e \in E_\lozenge$ is defined by

$$D^t V_e(u) = 0, \quad t \in \{(0,0),(1,0),(0,1)\}, \ u \in V_\lozenge, \tag{11}$$

$$D_c^{(1,1)} V_e(v_{c,1}) = \delta_{c,e}, \quad c = [v_{c,1}, v_{c,2}] \in E_\lozenge, \tag{12}$$

where

$$D^t f(x, y) = D_x^{t_1} D_y^{t_2} f(x, y),$$

$$D_e^t f(x, y) = (D_{v_e - v_{e,1}})^{t_1} (D_{v_{e,2} - v_{e,1}})^{t_2} f(x, y)$$

for $t = (t_1, t_2) \in \mathbb{Z}_+^2$. Here and throughout, as usual, the symbols $\delta_{t,s}$ and $\delta_{u,v}$ are the Kronecker delta.

We now begin to determine the B-net ordinates α_i, $i = 1, 2, 3$, η, and β as shown in Fig. 2 of [22] and to give all B-net ordinates of the edge spline V_e. We divide our discussion into four cases.

(1) *The vertex spline* $V_{v,(0,0)}(x, y)$.

In this case, condition (9) can be written as

$$V_{v,(0,0)}(v_1) = 1,$$

$$D_x V_{v,(0,0)}(v_1) = 0,$$

$$D_y V_{v,(0,0)}(v_1) = 0.$$

By using (3)–(5), the above equations are equivalent to

$$\alpha_1 = 1,$$

$$(y_2 - y_O)\alpha_1 + (y_O - y_1)\alpha_3 + (y_1 - y_2)\alpha = 0,$$

$$(x_O - x_2)\alpha_1 + (x_1 - x_O)\alpha_3 + (x_2 - x_1)\alpha = 0$$

if we restrict $V_{v,(0,0)}(x, y)$ on the triangle $\triangle v_1 v_2 O$, and are equivalent to

$$\alpha_1 = 1,$$

$$(y_O - y_4)\alpha_1 + (y_4 - y_1)\alpha + (y_1 - y_O)\alpha_2 = 0,$$

$$(x_4 - x_O)\alpha_1 + (x_1 - x_4)\alpha + (x_O - x_1)\alpha_2 = 0$$

if we restrict $V_{v,(0,0)}(x, y)$ on the triangle $\triangle v_1 O v_4$, where $\alpha = (\alpha_2 + k\alpha_3)/(1 + k)$ is the B-net ordinate as shown in Fig. 2 of [22] and (x_O, y_O) and (x_i, y_i) are coordinates of vertex O and vertices v_i, $i = 1, \ldots, 4$, respectively. Hence, we have

$$\alpha = \alpha_i = 1, \quad i = 1, 2, 3.$$

We now consider the condition (10) for $e = [v_1, v_2]$. Without loss of generality, we assume that the area of the triangle $\triangle v_1 v_2 O$ as shown in Fig. 2 of [22] is larger than that of the triangle which shares the common edge $[v_1, v_2]$ with $\triangle v_1 v_2 O$. Then the vertex v_e can be chosen as O and the condition (10) is clarified as

$$D_{O - v_1} D_{v_2 - v_1} V_{v,(0,0)}(v_1) = 0,$$

or equivalently

$$(x_2 - x_1)(x_O - x_1)D_{xx}V_{v,(0,0)}(v_1) + (x_O - x_1)(y_2 - y_1)D_{xy}V_{v,(0,0)}(v_1)$$

$$+ (x_2 - x_1)(y_O - y_1)D_{xy}V_{v,(0,0)}(v_1) + (y_O - y_1)(y_2 - y_1)D_{yy}V_{v,(0,0)}(v_1) = 0.$$

By using (6)–(8), the above equation can be transformed into
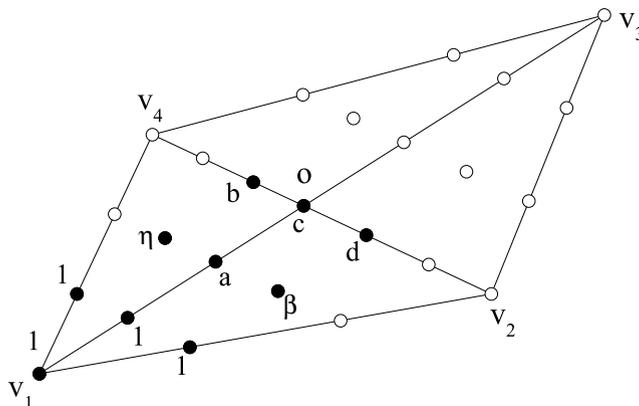
$$b_{300} - b_{201} - b_{210} + b_{111} = 0,$$

Fig. 1. B-net ordinates of $V_{v,(0,0)}$: $a = (\eta + k\beta)/(1 + k)$, $b = \eta/(1 + h)$, $c = (\eta + k\beta)/((1 + k)(1 + h))$ and $d = \eta/(1 + h)$ with $k = |v_4 - O|/|O - v_2|$ and $h = |v_1 - O|/|O - v_3|$, where $\beta = 1$ (or $\eta = 1$) if the area of $\triangle v_1v_2O$ (or $\triangle v_1Ov_4$) is larger then that of the triangle sharing the common edge $[v_1, v_2]$ (or $[v_1, v_4]$) with $\triangle v_1v_2O$ (or $\triangle v_1Ov_4$), otherwise β (or η) is determined by smoothness conditions along $[v_1, v_2]$ (or $[v_1, v_4]$). In addition, other B-net ordinates on all domain points "○" are vanished.

or equivalently,

$$\alpha_1 - \alpha - \alpha_3 + \beta = 0,$$

and thus, $\beta = 1$. Similarly, we can also determine η. The support of the vertex spline $V_{v,(0,0)}(x, y)$ and the corresponding B-net ordinates for this case are displayed in Fig. 1.

(2) *The vertex spline* $V_{v,(1,0)}(x, y)$.
In this case, condition (9) can be written as

$$V_{v,(1,0)}(v_1) = 0,$$

$$D_x V_{v,(1,0)}(v_1) = 1,$$

$$D_y V_{v,(1,0)}(v_1) = 0.$$

By using (3)–(5), the above equations are equivalent to

$$\alpha_1 = 0,$$

$$(y_2 - y_O)\alpha_1 + (y_O - y_1)\alpha_3 + (y_1 - y_2)\alpha = \frac{2S_{\triangle v_1v_2O}}{3},$$

$$(x_O - x_2)\alpha_1 + (x_1 - x_O)\alpha_3 + (x_2 - x_1)\alpha = 0$$

if we restrict $V_{v,(1,0)}(x, y)$ on the triangle $\triangle v_1v_2O$, and are equivalent to

$$\alpha_1 = 0,$$

$$(y_O - y_4)\alpha_1 + (y_4 - y_1)\alpha + (y_1 - y_O)\alpha_2 = \frac{2S_{\triangle v_1Ov_4}}{3},$$

$$(x_4 - x_O)\alpha_1 + (x_1 - x_4)\alpha + (x_O - x_1)\alpha_2 = 0$$
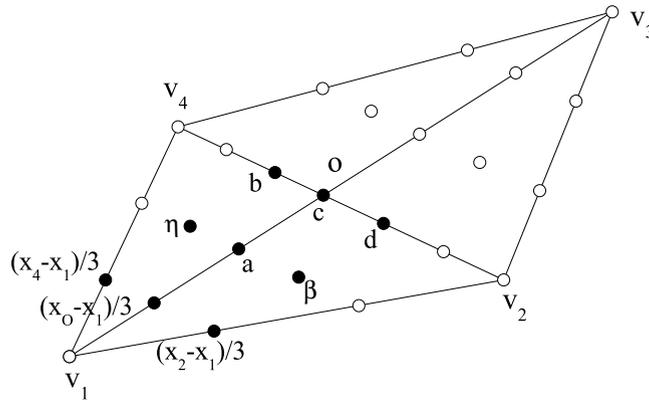
Fig. 2. B-net ordinates of $V_{v,(1,0)}$: $a = (\eta + k\beta)/(1 + k)$, $b = \eta/(1 + h)$, $c = (\eta + k\beta)/((1 + k)(1 + h))$ and $d = \eta/(1 + h)$ with $k = |v_4 - O|/|O - v_2|$ and $h = |v_1 - O|/|O - v_3|$, where $\beta = (x_2 + x_O - 2x_1)/3$ (or $\eta = (x_4 + x_O - 2x_1)/3$) if the area of $\triangle v_1 v_2 O$ (or $\triangle v_1 O v_4$) is larger then that of the triangle sharing the common edge $[v_1, v_2]$ (or $[v_1, v_4]$) with $\triangle v_1 v_2 O$ (or $\triangle v_1 O v_4$), otherwise β (or η) is determined by smoothness conditions along $[v_1, v_2]$ (or $[v_1, v_4]$). In addition, other B-net ordinates on all domain points "○" are vanished.

if we restrict $V_{v,(1,0)}(x, y)$ on the triangle $\triangle v_1 O v_4$, where $\alpha = (\alpha_2 + k\alpha_3)/(1 + k)$ is the B-net ordinate as shown in Fig. 2 of [22]. Hence, we have

$$\alpha_1 = 0, \quad \alpha_2 = \frac{x_4 - x_1}{3}, \quad \alpha_3 = \frac{x_2 - x_1}{3}, \quad \alpha = \frac{x_O - x_1}{3}.$$

We now consider condition (10) for $e = [v_1, v_2]$. Without loss of generality, we assume that the area of the triangle $\triangle v_1 v_2 O$ in Fig. 2 in [22] is larger than that of the triangle which shares the common edge $[v_1, v_2]$ with $\triangle v_1 v_2 O$. Then the vertex v_e can be chosen as O and condition (10) is clarified as

$$D_{O-v_1} D_{v_2-v_1} V_{v,(1,0)}(v_1) = 0$$

or

$$(x_2 - x_1)(x_O - x_1)D_{xx}V_{v,(1,0)}(v_1) + (x_O - x_1)(y_2 - y_1)D_{xy}V_{v,(1,0)}(v_1)$$

$$+ (x_2 - x_1)(y_O - y_1)D_{xy}V_{v,(1,0)}(v_1) + (y_O - y_1)(y_2 - y_1)D_{yy}V_{v,(1,0)}(v_1) = 0.$$

By using (6)–(8), the above equation can be transformed into

$$b_{300} - b_{201} - b_{210} + b_{111} = 0$$

or equivalently

$$\alpha_1 - \alpha - \alpha_3 + \beta = 0,$$

and thus,

$$\beta = \frac{x_2 + x_O - 2x_1}{3}.$$

Similarly, we can also determine η. The support of the vertex spline $V_{v,(1,0)}(x, y)$ and the corresponding B-net ordinates for this case are displayed in Fig. 2.

(3) *The vertex spline* $V_{v,(0,1)}(x, y)$.

In this case, condition (9) can be written as

$$V_{v,(0,1)}(v_1) = 0,$$

$$D_x V_{v,(0,1)}(v_1) = 0,$$

$$D_y V_{v,(0,1)}(v_1) = 1.$$

By using (3)–(5), the above equations are equivalent to

$$\alpha_1 = 0,$$

$$(y_2 - y_O)\alpha_1 + (y_O - y_1)\alpha_3 + (y_1 - y_2)\alpha = 0,$$

$$(x_O - x_2)\alpha_1 + (x_1 - x_O)\alpha_3 + (x_2 - x_1)\alpha = \frac{2S_{\triangle v_1 v_2 O}}{3}$$

if $V_{v,(0,1)}(x, y)$ is restricted on the triangle $\triangle v_1 v_2 O$, and are equivalent to

$$\alpha_1 = 0,$$

$$(y_O - y_4)\alpha_1 + (y_4 - y_1)\alpha + (y_1 - y_O)\alpha_2 = 0,$$

$$(x_4 - x_O)\alpha_1 + (x_1 - x_4)\alpha + (x_O - x_1)\alpha_2 = \frac{2S_{\triangle v_1 O v_4}}{3}$$

if $V_{v,(0,1)}(x, y)$ is restricted on the triangle $\triangle v_1 O v_4$, where $b = (\alpha_2 + k\alpha_3)/(1+k)$ is the B-net ordinate as shown in Fig. 2 of [22]. Hence, we have

$$\alpha_1 = 0, \quad \alpha_2 = \frac{y_4 - y_1}{3}, \quad \alpha_3 = \frac{y_2 - y_1}{3}, \quad \alpha = \frac{y_O - y_1}{3}.$$

We now consider condition (10) for $e = [v_1, v_2]$. Without loss of generality, we assume that the area of the triangle $\triangle v_1 v_2 O$ in Fig. 2 of [22] is larger than that of the triangle which shares the common edge $[v_1, v_2]$ with $\triangle v_1 v_2 O$. Then the vertex v_e can be chosen as O and the condition (10) is clarified as

$$D_{O-v_1} D_{v_2-v_1} V_{v,(0,1)}(v_1) = 0$$

or

$$(x_2 - x_1)(x_O - x_1)D_{xx}V_{v,(0,1)}(v_1) + (x_O - x_1)(y_2 - y_1)D_{xy}V_{v,(0,1)}(v_1)$$

$$+ (x_2 - x_1)(y_O - y_1)D_{xy}V_{v,(0,1)}(v_1) + (y_O - y_1)(y_2 - y_1)D_{yy}V_{v,(0,1)}(v_1) = 0.$$

By using (6)–(8), the above equation can be transformed into

$$b_{300} - b_{201} - b_{210} + b_{111} = 0,$$

or equivalently,

$$\alpha_1 - \alpha - \alpha_3 + \beta = 0,$$

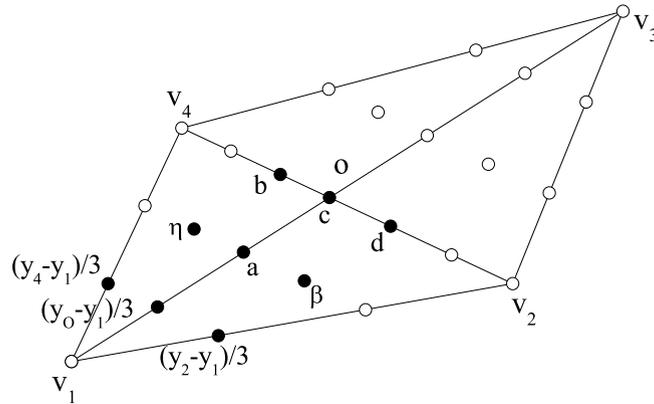and thus, we obtain

$$\beta = \frac{y_2 + y_O - 2y_1}{3}.$$

Fig. 3. B-net ordinates of $V_{v,(0,1)}$: $a = (\eta + k\beta)/(1 + k)$, $b = \eta/(1 + h)$, $c = (\eta + k\beta)/((1 + k)(1 + h))$ and $d = \eta/(1 + h)$ with $k = |v_4 - O|/|O - v_2|$ and $h = |v_1 - O|/|O - v_3|$, where $\beta = (y_2 + y_O - 2y_1)/3$ (or $\eta = (y_4 + y_O - 2y_1)/3$) if the area of $\triangle v_1 v_2 O$ (or $\triangle v_1 O v_4$) is larger then that of the triangle sharing the common edge $[v_1, v_2]$ (or $[v_1, v_4]$) with $\triangle v_1 v_2 O$ (or $\triangle v_1 O v_4$), otherwise β (or η) is determined by smoothness conditions along $[v_1, v_2]$ (or $[v_1, v_4]$). In addition, other B-net ordinates on all domain points "∘" are vanished.

Similarly, we can also determine η. The support of the vertex spline $V_{v,(0,1)}(x, y)$ and the related B-net ordinates for this case are displayed in Fig. 3.

(4) *The edge spline $V_e(x, y)$.*

For any triangle $[v_0^{[l]}, v_1^{[l]}, v_2^{[l]}] \in \Diamond$ with $v_0^{[l]} \in V_\Diamond$ and $v_j^{[l]} = (x_j^{[l]}, y_j^{[l]})$, $j = 0, 1, 2$, condition (11) can be written as

$$V_e(v_0^{[l]}) = 0,$$

$$D_x V_e(v_0^{[l]}) = 0,$$

$$D_y V_e(v_0^{[l]}) = 0.$$

It follows from (3)–(5) that the above equations are equivalent to

$$b_{300}^{[l]} = 0,$$

$$(y_1^{[l]} - y_2^{[l]})b_{300}^{[l]} + (y_2^{[l]} - y_0^{[l]})b_{210}^{[l]} + (y_0^{[l]} - y_1^{[l]})b_{201}^{[l]} = 0,$$

$$(x_2^{[l]} - x_1^{[l]})b_{300}^{[l]} + (x_0^{[l]} - x_2^{[l]})b_{210}^{[l]} + (x_1^{[l]} - x_0^{[l]})b_{201}^{[l]} = 0.$$

Hence,

$$b_{300}^{[l]} = 0, \quad b_{210}^{[l]} = 0, \quad b_{201}^{[l]} = 0.$$

In other words, for any vertex $v \in V_\Diamond$, all the B-net ordinates related to the domain points in the disk $D_1(v)$ are vanished.

We now consider condition (12) for any $c = [v_{c,1}, v_{c,2}] \in E_\Diamond$. Clearly, there are two triangles in $\Diamond$ sharing c. Let us choose the one which has the larger area if possible. Otherwise we choose any one of them. Denote it by $[v_{c,1}, v_{c,2}, v_c]$. If c is a boundary edge then there is one triangle $[v_{c,1}, v_{c,2}, v_c]$
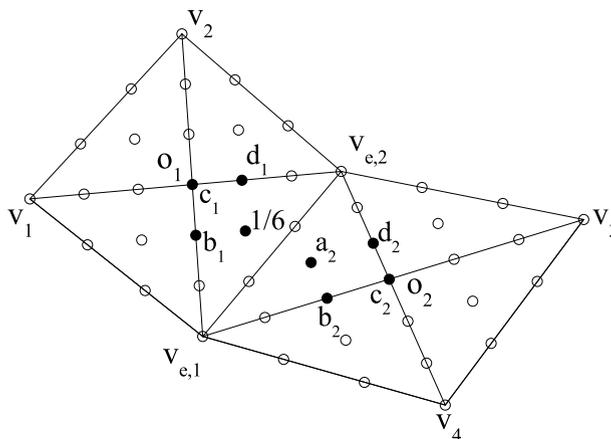
Fig. 4. B-net ordinates of V_e: $b_1 = k/(6(1+k))$, $c_1 = k/(6(1+k)(1+h))$, $d_1 = 1/(6(1+h))$, $a_2 = -\frac{1}{6} S_{\triangle_{O_2 v_{e,1} v_{e,2}}} / S_{\triangle_{O_1 v_{e,1} v_{e,2}}}$, $b_2 = a_2/(1+h')$, $c_2 = a_2/((1+k')(1+h'))$ and $d_2 = a_2/(1+k')$ with $k = |O_1 v_1|/|O_1 v_{e,2}|$, $h = |O_1 v_{e,1}|/|O_1 v_2|$, $k' = |O_2 v_{e,2}|/|O_2 v_4|$ and $h' = |O_2 v_{e,1}|/|O_2 v_3|$, while other B-net ordinates on all domain points "∘" are vanished.

containing edge c. Let $v_{c,1} = (x_{c,1}, y_{c,1})$, $v_{c,2} = (x_{c,2}, y_{c,2})$ and $v_c = (x_c, y_c)$. Then condition (12) is clarified as

$$D_{v_e - v_{c,1}} D_{v_{c,2} - v_{c,1}} V_e(v_{c,1}) = \delta_{c,e},$$

or equivalently,

$$(x_{c,2} - x_{c,1})(x_c - x_{c,1}) D_{xx} V_e(v_{c,1}) + (x_c - x_{c,1})(y_{c,2} - y_{c,1}) D_{xy} V_e(v_{c,1})$$

$$+ (x_{c,2} - x_{c,1})(y_c - y_{c,1}) D_{xy} V_e(v_{c,1}) + (y_c - y_{c,1})(y_{c,2} - y_{c,1}) D_{yy} V_e(v_{c,1}) = \delta_{c,e}.$$

By using (6)–(8), the above equation can be transformed into

$$b_{300} - b_{201} - b_{210} + b_{111} = \frac{1}{6} \delta_{c,e},$$

where b_{ijk}, $i + j + k = 3$ is the B-net ordinate of $V_e(x, y)$ with respect to the triangle $[v_{c,1}, v_{c,2}, v_c]$. Noticing that $b_{300} = 0$, $b_{210} = 0$, $b_{201} = 0$, we have

$$b_{111} = \frac{1}{6} \delta_{c,e}.$$

Finally, the remainder of the B-net ordinates on the whole triangulation $\triangle$ can be explicitly determined by using smoothness conditions. The support of the edge spline $V_e(x, y)$ and the corresponding B-net ordinates are shown in Fig. 4.

Acknowledgements

References

[1] P. Alfeld, A bivariate C^2 Clough–Tocher scheme, Comput. Aided Geom. Des. 1 (1984) 257–267.

[2] P. Alfeld, Bivariate splines and the four color problem, manuscript, 2000, http://www.math.utah.edu/alfeld/talks/S13/4CMP/s1.html

[3] P. Alfeld, B. Piper, L.L. Schumaker, Minimally supported bases for spaces of bivariate piecewise polynomials of smoothness r and degree $k \geqslant 4r + 1$, Comput. Aided Geom. Des. 4 (1987) 105–123.

[4] P. Alfeld, B. Piper, L.L. Schumaker, An explicit basis for C^1 quartic bivariate splines, SIAM J. Numer. Anal. 24 (1987) 891–911.

[5] P. Alfeld, L.L. Schumaker, The dimension of bivariate spline spaces of smoothness r for degree $d \geqslant 4r + 1$, Const. Approx. 3 (1987) 189–197.

[6] Y.S. Chou, L.Y. Su, R.H. Wang, The dimensions of bivariate spline spaces over triangulations, Int. Ser. Numer. Math., Vol. 75, Birkhauser, Basel, 1985, pp. 71–83.

[7] C.K. Chui, T.X. He, Bivariate C^1 quadratic finite elements and vertex splines, Math. Comp. 54 (1990) 169–187.

[8] C.K. Chui, D. Hong, Construction of the local C^1 quartic spline elements for optimal order approximation, Math. Comp. 65 (1996) 85–98.

[9] P.G. Ciarlet, Sur l'element de Clough et Tocher, Rev. Francaise Auto. Int. Rech. Oper. 8 (1974) 19–27.

[10] R.W. Clough, J.L. Tocher, Finite element stiffness matrices for analysis of plates in bending, Proceedings of the Conference on Matrix Methods in Structural Mechanics, Wright-Patterson A.F.B., OH, 1965.

[11] D. Diener, Instability in the dimension of spaces of bivariate piecewise polynomials of degree $2r$ and smoothness r, SIAM J. Numer. Anal. 27 (1990) 543–551.

[12] D. Diener, Geometry dependence of the dimension of spaces of piecewise polynomials on rectilinear partitions, Comput. Aided Geom. Des. 14 (1997) 43–50.

[13] G. Farin, Triangular Bernstein-Bézier patches, Comput. Aided Geom. Des. 3 (1986) 87–127.

[14] B. Fraejis de Veubek, Bending and stretching of plates, Proceedings of the Conference on Matrix Methods in Structural Mechanics, Air Force Institute of Technology, Dayton, OH, 1965.

[15] J.B. Gao, On the dimension of the bivariate spline spaces $S^r_{3r}(\triangle^*)$, J. Math. Res. Exp. 14 (1994) 367–378.

[16] G. Heindl, Interpolation and approximation by piecewise quadratic C^1-functions of two variables, in: W. Schempp, K. Zeller (Eds.), Multivariate Approximation Theory, Birkhäuser, Basel, 1979, pp. 146–161.

[17] D. Hong, Spaces of bivariate spline functions over triangulations, Approx. Theory Appl. 7 (1991) 56–75.

[18] D. Hong, H.-W. Liu, Some new formulation of smoothness conditions and conformality conditions for bivariate cubic splines, Comput. Math. Appl. 40 (2000) 117–125.

[19] D. Hong, L.L. Schumaker, Surface compression using a basis of C^1 cubic bivariate spline spaces, Adv. Comput. Appl. Math., to appear.

[20] M.J. Lai, Approximation order from bivariate C^1-cubics on a four-directional mesh is full, Comput. Aided Geom. Des. 11 (1994) 215–223.

[21] M.J. Lai, On C^2 quintic spline functions over triangulations of Powell–Sabin's type, J. Comput. Appl. Math. 73 (1996) 135–155.

[22] M.J. Lai, Scattered data interpolation and approximation using bivariate C^1 piecewise cubic polynomials, Comput. Aided Geom. Des. 13 (1996) 81–88.

[23] M.J. Lai, L.L. Schumaker, On the approximation power of splines on triangulated quadrangulations, SIAM J. Num. Anal. 36 (1999) 143–159.

[24] M.J. Lai, P. Wenston, Bivariate spline method for numerical solution of the steady state Navier–Stokes equations over polygons in stream function formulation, Numer. Methods for PDE's 16 (2000) 147–183.

[25] H.-W. Liu, An integral representation of bivariate splines and the dimension of quadratic spline spaces over stratified triangulations, Acta Math. Sinica 37 (1994) 534–543.

[26] H.-W. Liu, The dimension of cubic spline space over stratified triangulations, J. Math. Res. Exposition 16 (1996) 199–208.

[27] H.-W. Liu, D. Hong, Some smoothness conditions and conformality conditions for bivariate quartic and quintic splines, CALCOLO 36 (1999) 43–61.

[28] H.-W. Liu, D. Hong, The bivariate C^1 cubic spline space over even stratified triangulations, J. Comput. Anal. Appl. 4 (2002) 19–35.

[29] J. Morgan, R. Scott, A nodal basis for C^1 piecewise polynomials of degree $n \geqslant 5$, Math. Comp. 29 (1975) 736–740.

[30] J. Morgan, R. Scott, The dimension of piecewise polynomial, manuscript, 1975.

[31] P. Percell, On cubic and quartic Clough–Tocher finite elements, SIAM J. Numer. Anal. 13 (1976) 100–103.

[32] M.J.D. Powell, M.A. Sabin, Piecewise quadratic approximations on triangles, ACM Trans. Math. Software 3 (1977) 316–325.

[33] P. Sablonniere, Composite finite elements of class C^2, in: C.K. Chui, L.L. Schumaker, F.I. Utreras (Eds.), Topics in Multivariate Approximation, Academic Press, New York, 1987, pp. 207–217.

[34] G. Sander, Bornes superieures et inferieures dans l'analyse matricille des plaques en flexion-torsion, Bull. Soc. r. Sci. Liege 33 (1964) 456–494.

[35] L.L. Schumaker, On the dimension of spaces of piecewise polynomials in two variables, in: W. Schempp, K. Zeller (Eds.), Multivariable Approximation Theory, Birkhäuser, Basel, 1979, pp. 396–412.

[36] L.L. Schumaker, On super splines and finite elements, SIAM J. Numer. Anal. 26 (1989) 997–1005.

[37] X.-Q. Shi, The singularity of Morgan–Scott triangulation, Comput. Aided Geom. Des. 8 (1991) 201–206.

[38] G. Strang, Piecewise polynomials and the finite elements method, Bull. Amer. Math. Soc. 79 (1973) 736–740.

[39] G. Strang, The dimension of piecewise polynomials, and one-sided approximation, in: Proceedings of the Conference on Numerical Solution of Differential Equations, Dundee, Lecture Notes in Mathematics, Vol. 365, Springer, Berlin, New York, 1974, pp. 144–152.

[40] T. Wang, A C^2-quintic spline interpolation scheme on triangulation, Comput. Aided Geom. Des. 9 (1992) 379–386.

[41] M.D. Ye, Some problems for the bivariate C^1 cubic splines, Approx. Theory Appl. 4 (1988) 1–11.

[42] X.-W. Zhang, Y. Lin, The dimension of spline space $S_3^1(\triangle)$ on a type of triangulation, Appl. Math. Mech. 21 (2000) 783–791.