Gambler's Ruin Problem

Suppose a gambler starts a sequence of games with an initial amount of i dollars. On each play she either wins $1 with probability p or loses $1 with probability $q = 1 - p$. She plays until she's broke or has N dollars (that is, wins $N - i$ dollars). Find P_i, the probability that she wins $N - i$ dollars.

Solution. Conditioning on the first play, we have $P_i = p\,P_{i+1} + q\,P_{i-1}$. Since we can write $P_i = pP_i + qP_i$, we obtain

$$p(P_{i+1} - P_i) = q(P_i - P_{i-1}),$$

which implies

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}).$$

Letting $D_k = P_k - P_{k-1}$, we have the recursive relationship

$$D_{i+1} = \frac{q}{p}D_i \text{ for } i = 1, 2, \ldots \text{ with } D_1 = P_1.$$

Hence

$$D_n = \left(\frac{q}{p}\right)^{n-1} P_1.$$

Now note that

$$\begin{aligned} P_i &= P_1 + (P_2 - P_1) + \ldots + (P_i - P_{i-1}) \\ &= \sum_{k=1}^{i} D_k \\ &= \sum_{k=1}^{i} \left(\frac{q}{p}\right)^{k-1} P_1 \\ &= P_1 \cdot \frac{1-(q/p)^i}{1-(q/p)} \end{aligned}$$

provided $q \neq p$. If $q = p = \frac{1}{2}$, then $D_k = P_1$ and thus

$$P_i = \sum_{k=1}^{i} D_k = i\,P_1.$$

Furthermore, since $P_N = 1$, we find $P_1 = \frac{1-(q/p)}{1-(q/p)^N}$ when $p \neq q$ and $P_1 = \frac{1}{N}$ when $p = q$. Therefore,

$$P_i = \frac{1-(q/p)^i}{1-(q/p)^N} \text{ when } p \neq \tfrac{1}{2} \text{ and} \tag{1}$$

$$P_i = \frac{i}{N} \text{ when } p = \tfrac{1}{2}. \tag{2}$$

Example. A gambler starts a sequence of games with an initial amount of $i = 25$ dollars. On each play she either wins \$1 with probability $p = .48$ or loses \$1 with probability $q = 1 - p = .52$. She plays until she's broke or has $N = 30$ dollars (that is, wins $N - i = 5$ dollars).

(i) Find P_{25}, the probability that she wins $30 - 25 = 5$ dollars.

Using the formula in (1) we have

$$P_{25} = \frac{1-(.52/.48)^{25}}{1-(.52/.48)^{30}} = .63732$$

Although the probability of winning here is greater than $1/2$, why is this not a recommended betting scheme?

(ii) Find her expected winnings per sequence of games.

$$E(W) = -\$25\,(.36268) + \$5(.63732) = -\$5.88$$

The following table gives P_i for various $N - i$ values when $i = 25$ and $p = .48$.

$N - 25$	P_{25}
1	.91211
3	.76113
5	.63732
7	.53515
9	.45041
13	.32082
15	.27135

Expected winnings, $E(W)$, per sequence when $q > 1/2$

Assume $q > 1/2$. Then, the odds for losing a game, $r = q/p > 1$. We also assume that $N > i$.

$$\begin{aligned} E(W) &= (N-i)P_i + (-i)(1-P_i) \\ &= NP_i - iP_i - i + iP_i \\ &= NP_i - i \end{aligned}$$

We want to show that $E(W) = NP_i - i < 0$.
First, let $N = i + k$ where $k \in \mathcal{N}$. Then $P_i = \frac{r^i - 1}{r^{i+k} - 1}$

Hence $\quad E(W) < 0 \quad$ iff $\quad NP_i - i < 0$

$\quad$ iff $\quad (i + k)P_i - i < 0$

iff $\frac{(i+k)}{k} P_i - \frac{i}{k} < 0$

iff $\frac{i}{k} > \frac{(i+k)}{k} P_i$

iff $\frac{i}{k}(1 - P_i) > P_i$

iff $\frac{i}{k} > \frac{P_i}{(1-P_i)}$

iff $\frac{i}{k} > \frac{\frac{r^i-1}{r^{i+k}-1}}{(1-\frac{r^i-1}{r^{i+k}-1})}$

iff $\frac{i}{k} > \frac{r^i-1}{r^{i+k}-r^i}$

iff $\frac{i}{k} > \frac{r^i-1}{r^i(r^k-1)}$ (1).

We prove (1) using induction on i.

For $i = 1,$ we have $\frac{1}{k} > \frac{1}{r+r^2+\ldots+r^k}$ since $r > 1.$
But

$$\frac{1}{r+r^2+\ldots+r^k} = \frac{r-1}{r(r^k-1)} \text{ and hence } \frac{1}{k} > \frac{r-1}{r(r^k-1)}. \qquad (2)$$

Now assume $\frac{i}{k} > \frac{r^i-1}{r^i(r^k-1)}$ for arbitrary positive integer i.
Then

$$\frac{i+1}{k} = \frac{i}{k} + \frac{1}{k}$$

$$> \frac{r^i-1}{r^i(r^k-1)} + \frac{1}{k} \text{ (by the induction hypothesis)}$$

$$> \frac{r^i-1}{r^i(r^k-1)} + \frac{r-1}{r(r^k-1)} \text{ (by (2))}$$

$$= \frac{r(r^i-1)+r^i(r-1)}{r^{i+1}(r^k-1)}$$

$$= \frac{r^{i+1}-1-(r-1)+r^i(r-1)}{r^{i+1}(r^k-1)}$$

$$= \frac{r^{i+1}-1+(r-1)(r^i-1)}{r^{i+1}(r^k-1)}$$

$$> \frac{r^{i+1}-1}{r^{i+1}(r^k-1)} \quad \text{since } r > 1 \qquad \square$$

Example. Suppose you start with two dollars and on each play you either win a dollar with probability .4 or lose a dollar with probability .6. You decide at the beginning that you will quit if you are ever ahead by $2 or if you go broke. Find the probability that you will end up winning $2.

Here the states will be the money at hand $-\ 0, 1, 2, 3,$ or 4 dollars. Your initial state probability vector is $[0,\ 0,\ 1,\ 0,\ 0]$ since you start with 2 dollars. The transition probability matrix $\mathbb{P}$ is

$$\mathbb{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 & 0 \\ 0 & .6 & 0 & .4 & 0 \\ 0 & 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$P(\text{end up winning \$2}) = \frac{1-(q/p)^i}{1-(q/p)^N} = \frac{1-(.6/.4)^2}{1-(.6/.4)^4} = \frac{1-9/4}{1-81/16} = \frac{20}{65} = \frac{4}{13}.$$

Alternatively, note

$$[0,\ 0,\ 1,\ 0,\ 0]\mathbb{P} = [0\ .6\ 0\ .4\ 0]$$

and

$$[0,\ 0,\ 1,\ 0,\ 0]\mathbb{P}^2 = [.36\ 0\ .48\ 0\ .16].$$

After two plays you have probability .16 of winning $2 and probability .36 of being broke In fact,

$$P(\text{end up winning \$2}) = \frac{16}{16+36} = \frac{4}{13}.$$

Why? Because the possible winnings are double the initial stake.
In general, if initial stake is i and one quits upon winning $2i$ or going broke,
then $P(\text{end up winning } i) = P_i = \frac{p^i}{p^i+q^i} = \frac{p^i}{p^i+(1-p)^i}$.

For example if $p = .49$, then $P_i = .49^i/(.49^i + .51^i)$.

i	P_i
1	.49
5	.45016
10	.4013
15	.35433
20	.31
25	.26892
27	.25348

Example. A gambler starts a sequence of games with an initial amount of $i = 25$ dollars. On each play she either wins $1 with probability $p = .48$ or loses $1 with probability $q = 1 - p = .52$. She plays until she's broke or has $N = 40$ dollars (that is, wins $N - i = 15$ dollars).

(i) Find P_{25}, the probability that she wins $40 - 25 = 15$ dollars.

Using the formula in (1) we have

$$P_{25} = \frac{1-(.52/.48)^{25}}{1-(.52/.48)^{40}} = .2713506671$$

(ii) What is her expected winnings per sequence of games?

$$E(W) = (1 - .2713506671)(-25) + (.2713506671)(15) \approx -\$\,14.15$$