A Note on Counting Subsets of Restricted Size

Dennis P. Walsh
Middle Tennessee State University
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Let $[n]$ denote the set of the first n positive integers. Since the number of size k subsets of $[n]$ is $\binom{n}{k}$, the number of subsets of $[n]$ of at most size k is $\sum_{j=0}^{k}\binom{n}{k}$. These numbers form sequence $A008949$ in the *On-Line Encyclopedia of Integer Sequence* (available at http:oeis.org/A008949).

We will provide an alternative formula for $T(n,k)$, the number of subsets of $[n]$ of at most size k. We first note that $T(n,k)$ is also equal to the number of subsets of $[n]$ of at least size $(n-k)$ since every subset of size j has a unique complement of size $(n-j)$.

Now note that any subset of at least size $(n-k)$ must contain $(n-k)$ smallest elements. Let m denote the largest of those $(n-k)$ smallest elements, and thus m takes on any value from $(n-k)$ to n. The remaining $(n-m)$ elements larger than m may or may not be in any particular subset. Hence, to construct a subset of size at least $(n-k)$ when given a specific m, we first select $n-k-1$ elements from $[m-1]$, and then we decide if any of the remaining elements from $(m+1)$ to n will be in the subset. The number of ways to perform such a construction is thus $\binom{m-1}{n-k-1}2^{n-m}$.

Therefore, summing over the possible values for m, we obtain

$$T(n,k)=\sum_{m=n-k}^{n}\binom{m-1}{n-k-1}2^{n-m} \qquad (1).$$

Letting $j=m-n+k$, we obtain the equivalent formula

$$T(n,k)=\sum_{j=0}^{k}\binom{n+j-k-1}{j}2^{k-j} \qquad (2).$$

The following combinatorial identity is proven

$$\sum_{j=0}^{k}\binom{n}{k}=\sum_{j=0}^{k}\binom{n+j-k-1}{j}2^{k-j} \qquad (3).$$

Example. Let $n=5$ and $k=3$. Then $\sum_{j=0}^{3}\binom{5}{k}=\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}=26$ and $\sum_{j=0}^{3}\binom{j+1}{j}2^{3-j}=\binom{1}{0}2^3+\binom{2}{1}2^2+\binom{3}{2}2+\binom{4}{1}1=26.$

We note that if one removes the factor (2^{k-j}) from the right side sum in identity (3) above, the "hockey-stick" formula for binomial coefficients comes into play, namely,

$$\sum_{j=0}^{k} \binom{n+j-k-1}{j} = \binom{n}{k}$$

For example, with $n = 5$ and $k = 3$, we obtain

$$\sum_{j=0}^{3} \binom{j+1}{j} = \binom{5}{3}.$$