L17: Rotational Dynamics

We have now been introduced to the fundamental angular variables: the angular displacement θ (*theta*), the angular velocity ω (*omega*), the angular acceleration α (*alpha*), and the angular version of force—the torque, τ (*tau*). We have seen that there are relatively straightforward relationships between the linear variables and the corresponding angular variables:

(linear quantity) = r (angular quantity)

For example, $x = r\,\theta$, $v_t = r\,\omega$, and $a_t = r\,\alpha$. The torque equation changes the format slightly: $\tau = r_{perp} F$.

We shall now go on to define the remaining angular quantities in rotational dynamics, and see that a simple translation exists between the linear relations and the corresponding angular relations. Indeed, some of the angular definitions were made simply to make the resulting relations among the angular variables *look like* the corresponding linear relations with which we are already familiar!

We start off with a discussion of inertia, both linear and rotational. We then move on to other relations, pretty much in the order in which we encountered the linear quantities. The last section of this lecture gives a table which summarizes the linear and angular quantities and relations among those quantities.

Inertia

We have actually already encountered an *inertia*, even though we didn't usually call it that. The ***translational inertia*** is just another name for ***mass***. We can define translational inertia, ***m***, as follows:

> **translational inertia, *m***, is a measure of the resistance of an object to a *change* in its motion

An object has a certain motion (*velocity*). We go to change the object's motion. How hard is it for us to ***change*** the object's motion? This is a measure of the object's *translational inertia* (*mass*).

For example, let's say that a large ball (maybe *3 ft* in diameter) is sliding quickly towards you across a very slippery (*frictionless*) surface. How would you feel if it were a giant Ping-Pong ball versus if it were a giant solid iron ball? You would probably feel much less threatened if it were a giant Ping-Pong ball (and rightly so!), since you know that it would be relatively easy to stop it, as opposed to the giant iron ball, which you would (*hopefully!*) be smart enough not to try to stop! The large Ping-Pong ball would not *resist* you changing its motion very much – it does not have a very large translational inertia. On the other hand, the large iron ball would resist you changing its motion in a major way – it would be *very* hard for you to stop it, as it has a correspondingly large *translational inertia* (*mass*).

Likewise, the ***rotational inertia*** (or the ***moment of inertia*** – the same thing, just a fancier name!), ***I*** , may be defined as follows:

> **rotational inertia, *I***, is a measure of the resistance of an object to a *change* in its *rotational* motion

An object has a certain rotational motion (*angular velocity*). We go to change the object's rotational motion. How hard is it for us to ***change*** this motion? This is a measure of the object's *rotational inertia* (or *moment of inertia*).

We can discuss *rotational inertia* using the giant Ping-Pong ball and iron ball used previously, only now the balls are mounted on vertical axes about which they can be rotated (just like giant globes). Both balls are spinning on their corresponding axes with the same angular velocity, ω. Our job is to stop the rotational motion. The giant Ping-Pong ball is relatively easy to stop – it does not resist this change in its motion — it has a relatively small rotational inertia. However, the giant iron ball is very difficult to stop – it offers a large resistance to a change in its rotational motion since it has a correspondingly large rotational inertia.

As we'll experience firsthand in the Problems Lab, the rotational inertia of an object or system depends not only on its *mass*, but also on the *distribution* of this mass about the axis of rotation. The further the mass is from the axis of rotation, the larger its rotational inertia, and the harder it is to change its rotational motion (starting it rotating, or stop its rotation, for example).

There are two ways to compute an object's rotational inertia, depending on how we want to treat the object (or system of objects). First, if the size of the object is small compared to the distance of the object from its axis of rotation, then we can treat it as a *point particle* (really small, relatively speaking). In this case, if the object's mass is m and its distance from the axis of rotation is R, then its rotational inertia is simply given by

$$I = m R^2 .$$

For example, the earth may often be treated as a uniform, solid sphere (it isn't really, but it's often a reasonable approximation to assume so). The rotation of the earth about its axis each day is an example of rotational motion. The size of the earth compared to the distance of its mass from the axis of rotation (both on the order of the radius of the earth!) is about the same, so it is a lousy approximation to use the equation given above to determine the earth's rotational inertia. However, if we want to consider the earth's rotation (or, more properly, its *revolution*) about the sun, then the size of the earth (its radius) is relatively small compared to its distance from the axis of revolution through the sun (the distance from the sun to the earth), so in this case it is a good approximation to use the equation given above to compute the earth's rotational inertia (or moment of inertia) about that axis (where R would be the distance from the sun to the earth).

So what do we do if we want the moment of inertia of the earth about its axis as it rotates each day? We could either calculate the moment of inertia of the object using some relatively advanced calculus (*not our job!*), or else we just look up the description of the object in a ***table*** of moments of inertia, such as that given in the next section of this lecture. In such a table, if we were to look up a solid, uniform sphere of mass M and radius R about an axis through its center, we would find that

$$I = \tfrac{2}{5} M R^2 .$$

In any problems, if you can treat the object as a point mass (that is, if it's small compared to its distance from the axis of rotation), then use $I = m R^2$ to compute its moment of inertia. Otherwise, you must look up an equation for I

in a table. Note that if you are considering a *system* of objects, then the moment if inertia of the system is simply the ***sum*** of the moments of inertia of the individual parts of the system.

Inertia Table

The following table gives some equations for the moments of inertia for the various mass distributions described. More complete tables of moments of inertia can be found in physics texts in the library or bookstore.

Description of Object	Moment of Inertia
cylindrical shell or ring of radius *R* and mass *M* about an axis through its center and along the axis of symmetry	$I = MR^2$
solid cylinder or disk of radius *R* and mass *M* about its axis of symmetry	$I = \frac{1}{2}MR^2$
thin-shelled hollow sphere of radius *R* and mass *M* about an axis through its center	$I = \frac{2}{3}MR^2$
solid sphere of radius *R* and mass *M* about an axis through its center	$I = \frac{2}{5}MR^2$
thin rod of length *L* and mass *M* about an axis perpendicular to the rod at one end	$I = \frac{1}{3}ML^2$
thin rod of length *L* and mass *M* about an axis perpendicular to the rod through its center	$I = \frac{1}{12}ML^2$

Solution to Example 17.1

You roll an orange and a cardboard tube from the inside of a roll of paper towels across the counter top in your kitchen. The orange has a radius of **6.5 cm** and a mass of **270 g**, while the tube has a mass of **57 g**, a length of **32 cm**, and a radius of **2.5 cm**. What are the moments of inertia of these two objects? Which would be harder to get rolling with a given angular speed across the counter top, starting from rest?

We start by assigning some symbols to the data we're given (subscripts: "o" = orange; "t" = tube):

$$R_o = 0.065 \text{ m} \quad M_o = 0.27 \text{ kg} \quad R_t = 0.025 \text{ m} \quad L_t = 0.32 \text{ m} \quad M_t = 0.057 \text{ kg}$$

These objects are to be rotated about axes through their centers, so the size of the objects and the distances of the mass in the objects from the axis of rotation are about the same. We can thus *not* use the equation $I = MR^2$, and so we need to look up the expressions for I in the table.

We can *approximately* model the orange as a solid sphere. We thus get that

$$\mathbf{I = \tfrac{2}{5}MR^2 = 4.6 \times 10^{-4}\ kg \cdot m^2}.$$

Note that the units of moment of inertia in the *MKS* system are just the units of *mass* times the units of *distance-squared*:**kg m²**.

We may treat the tube as a hollow cylindrical shell. Therefore,

$$\mathbf{I = MR^2 = 3.6 \times 10^{-5}\ kg \cdot m^2}.$$

Note that the length of the tube does not matter – *all* of the mass in the tube is at the same distance R from the axis of rotation, so the moment of inertia is just given by MR^2. Note also that the moments of inertia do not depend on the rotational motion of the objects – only on the axis of rotation that the object *will* have, regardless of whether it's rotating now or not!

Since the moment of inertia of the orange is greater than that of the tube, it follows that the orange will be harder to get rolling than the cardboard tube (as we would expect!).

Newton's 2nd Law Revisited

Newton's 2nd law as we stated it previously gives us a relationship between the *net force* acting on an object and the resulting *change* in the object's *motion* – in other words, its *acceleration*. Since the mass or *translational inertia* of the object gives us a measure of how hard it is to change its motion, it should be no surprise that it, too, plays a role in Newton's 2nd law:

$$\sum \vec{F} = m\vec{a}.$$

As far as *rotational* motion is concerned, a *torque* is that which causes a change in an object's *rotational* motion – that is, a torque causes an *angular* acceleration. Since the rotational inertia gives us a measure of how difficult it is to change an object's rotational motion, it should be no surprise that the ***rotational form of Newton's 2nd law*** is as follows:

$$\sum \vec{\tau} = I\vec{\alpha}.$$

Note the complete symmetry between the two equations above. We simply take Newton's 2nd law in the linear form, and replace each quantity with its corresponding angular counterpart:

$$\vec{F} \rightarrow \vec{\tau} \qquad m \rightarrow I \qquad \vec{a} \rightarrow \vec{\alpha}.$$

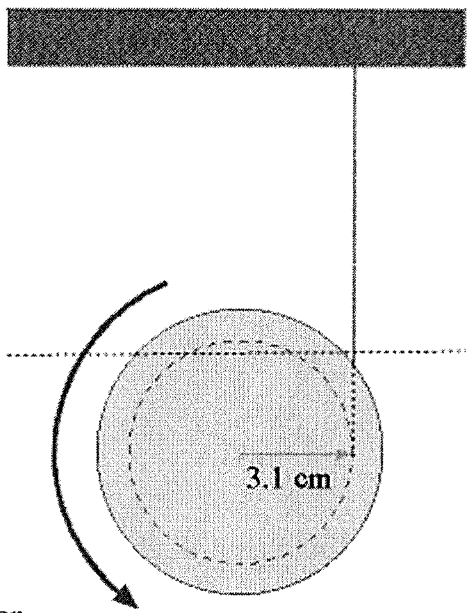
If an object is both rotating *and* translating (that is, it is rotating about an axis, while that axis itself is moving through space, like a ball rolling down an incline...), then we must use *both* equations above to completely describe the situation at hand. (Note that in all of the problems that we've dealt with so far, the object has always been *sliding* – across a surface, down an incline, *etc.* This was because we did not know how to deal with *rolling* motion. *This is about to change!*)

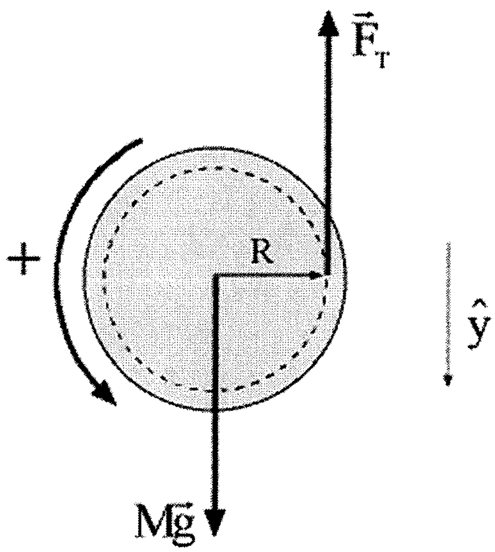
Solution to Example 17.2

A **120-g** yo-yo is unwinding its way down a vertical string of length **1.1 m**. Its angular acceleration about its axis of symmetry as it's spinning downward is **0.78 rad/s²**. The radius of the inner spool about which the string is wound is **3.1 cm**. (See L15 homework problem #6.) Find the yo-yo's moment of inertia about its axis of symmetry.

Data: $R = 0.031$ m $M = 0.12$ kg $\alpha = 0.78$ rad/s^2 $L = 1.1$ m

The set-up is shown at right. The vertical string supports the yo-yo as it unwinds, rotating about its axis of symmetry (the axis about which the yo-yo is the same in all directions; this axis is the axis through the yo-yo's center-of mass). As a result of its angular acceleration, the *CM* of the yo-yo also has a *linear* acceleration downward of magnitude $a_{cm} = \alpha R = 0.024\ m/s^2$. The key to the solution of this problem is to look at what's going on from both the linear ***and*** the angular points of view.





Consider the FBD for the yo-yo, shown at left. Applying Newton's 2nd law in linear form to the y-direction (downward, since this is the direction of acceleration), we get that

$$\sum F_y = Ma_y = Ma_{cm}$$

$$Mg - F_T = Ma_{cm}$$

$$F_T = M(g - a_{cm}) = 1.17\ N.$$

Taking positive torques to be *CCW* (in the direction of the angular acceleration), and the axis of rotation to be the axis through the center of the yo-yo (the axis of symmetry), we then get, applying Newton's 2nd law in rotational form, that

$$\sum \tau = I\alpha$$

$$F_T R = I\alpha$$

$$I = \frac{F_T R}{\alpha} = 0.047\ \mathrm{kg \cdot m^2}\ .$$

Note that the weight of the yo-yo did not exert a torque about the axis of rotation since it acts directly through that axis (its moment arm is zero; it does not tend to rotate the yo-yo one way or the other).

Again, the key to this solution was the application of Newton's 2nd law to the yo-yo in both the linear as well as the angular form, along with the relation between the linear and the angular motion as given by $a_{cm} = \alpha R$.

Momentum Revisited

In light of the discussion in the previous section, the following discussion should come as no surprise.... We are now about to address the topic of *angular momentum*. Before we do so, let's first remind ourselves of the definition of linear momentum.

The ***linear momentum, p,*** of an object is simply the product of the object's mass and its velocity:

$$\vec{p} = m\vec{v}.$$

We may restate this to say that the object's *linear* momentum is the product of the object's *translational inertia* and its *linear* velocity. You should be able to guess the rest! *Yes*, the object's ***angular momentum*** is simply the product of the object's *rotational inertia* and its *angular velocity*. The angular momentum is given the symbol ***L***:

$$\vec{L} = I\vec{\omega}.$$

Recall that Newton's 2nd law could be re-expressed in terms of the linear momentum of the object:

$$\vec{F}_{net} = \frac{\Delta\vec{p}}{\Delta t}.$$

Thus, a net force will change an object's motion or, equivalently, its linear momentum. Likewise, the rotational form of Newton's 2nd law can be expressed in terms of the object's *angular* momentum:

$$\vec{\tau}_{net} = \frac{\Delta\vec{L}}{\Delta t}.$$

Furthermore, we noted that, *if* the net force acting on an object were *zero*, then the change in linear momentum must be also be zero – meaning that the *initial* momentum must equal the *final* momentum. This then gave us the ***conservation of linear momentum***:

$$\vec{p}_i = \vec{p}_f.$$

Likewise, if the *net torque* acting on an object about some axis is zero, then the *change* in *angular momentum* must also be zero, telling us that, in this case, the *initial angular momentum must equal the final angular momentum.* This is the statement of the ***conservation of angular momentum***:

$$\vec{L}_i = \vec{L}_f .$$

This is an extremely important and interesting concept. We will experience firsthand some of the strange consequences of this principle in the Problems Lab. Typical *conservation of angular momentum* problems involve a system that is initially rotating at some *constant* angular velocity about some axis (meaning that the angular *acceleration*, and thus the *net torque* about that axis must be *zero*!). Something *internal* to the system then changes (if it's *internal*, then the net *external* torque must remain the same!), resulting in a change in the object's rotational motion. This is a definite clue that the *conservation of angular momentum* is the key to the problem's solution, as the next example shows.

Example 17.3

A cockroach is standing on the rim of a lazy susan (a flat, circular, platform which can rotate, usually used in kitchens or dining rooms) of radius **25 cm** and mass **85 g**. The cockroach suddenly notices a yummy bread crumb a bit further ahead on the rim, and starts walking along the rim to reach it. Walking with a constant speed, it takes the cockroach **3.2 s** to reach the bread crumb, which was a distance of **11 cm** along the rim of the lazy susan from the roach. While the cockroach is walking, it is found that the lazy susan is rotating in a direction *opposite* to the cockroach's motion with an angular velocity of **0.037 rad/s**. (a) What is the moment of inertia of the lazy susan? (b) What is the *equation* for the moment of inertia of the cockroach? (c) What is the total angular momentum of the system before the cockroach starts walking? (d) What is the total angular momentum of the system while the cockroach is walking? (e) How fast does the cockroach walk *relative to the lazy susan*? (f) What is the speed of a point on the rim of the lazy susan? (g) What is the cockroach's speed *relative to the earth*? (h) What is the magnitude of the lazy susan's angular momentum while the cockroach is walking? (i) What is the magnitude of the cockroach's angular momentum? (j) What is the mass of the cockroach?

Answers: (a) 0.0027 kg m^2 (b) m R^2 (c) 0 (d) 0 (e) 0.034 m/s (f) 0.0093 m/s (g) 0.025 m/s (h) 1.0×10^{-4} kg m^2/s (i) 1.0×10^{-4} kg m^2/s (j) 16 g

Solution

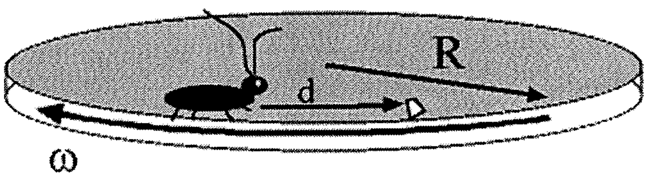
Solution to Example 17.3

A cockroach is standing on the rim of a lazy susan (a flat, circular, platform which can rotate, usually used in kitchens or dining rooms) of radius **25 cm** and mass **85 g.** The cockroach suddenly notices a yummy bread crumb a bit further ahead on the rim, and starts walking along the rim to reach it. Walking with a constant speed, it takes the cockroach **3.2 s** to reach the breadcrumb, which was a distance of **11 cm** along the rim of the lazy susan from the roach. While the cockroach is walking, it is found that the lazy susan is rotating in a direction *opposite* to the cockroach's motion with an angular velocity of **0.037 rad/s**.

Let's first define some symbols:

$R = 0.25$ m $M_{ls} = 0.085$ kg $t = 3.2$ s $d = 0.11$ m $\omega = 0.037$ rad/s

Now, let's proceed....



(a) What is the moment of inertia of the lazy susan?

We may treat the lazy susan as a disk of mass M_{ls} and radius R. Its moment of inertia is thus

$$I_{ls} = \tfrac{1}{2} M_{ls} R^2 = 0.0027 \text{ kg} \cdot \text{m}^2 .$$

(b) What is the *equation* for the moment of inertia of the cockroach?

We may reasonably assume that the size of the cockroach is much smaller than its distance from the axis of rotation (that is, the size of the cockroach is much smaller than R = *25 cm*!). We may thus treat it as a point mass, and write its rotational inertia as

$$I_{cr} = mR^2 .$$

(c) What is the total angular momentum of the system before the cockroach starts walking?

Since nothing is moving before the cockroach starts walking, it follows that the total angular momentum must equal *zero*. Also, since nothing is moving initially and, until the cockroach starts walking, everything *stays* this way, it follows that the net external torque acting on the system must also be *zero*.

(d) What is the total angular momentum of the system while the cockroach is walking?

The system that we are considering is the *lazy susan + cockroach* system. When the cockroach starts walking, it isn't because some *external* force came along to push it – rather, this was due to an *internal* force (some sort of cockroach force!). Thus, the net external torque must still be zero, and angular momentum must be *conserved*. This means that the total angular momentum *before* the cockroach started walking must equal the total angular momentum *after* it started walking. In other words, the total angular momentum of the system must still be *zero*, even after the roach starts walking!

(e) How fast does the cockroach walk *relative to the lazy susan*?

The roach walks a distance *d* in a time *t* across the surface of the lazy susan (the distance is measured along the surface of the lazy susan). Thus, the speed of the roach *relative to the lazy susan* must be

$$v_{ls} = \frac{d}{t} = 0.034\,\frac{m}{s}.$$

Note that we have put a subscript *"ls"* on the speed *v*. This is just to remind us that this speed is *relative to* the lazy susan.

(f) What is the speed of a point on the rim of the lazy susan?

We are told that the lazy susan is rotating with an angular velocity ω. The speed of a point on the rim of the lazy susan is thus (remember this one?)

$$V = \omega R = 0.0093\,\frac{m}{s}.$$

(We are using *V* for the speed of a point on the lazy susan, and *v* for the cockroach's speed....)

(g) What is the cockroach's speed *relative to the earth*?

If the lazy susan were not moving, then the cockroach's speed relative to the earth (or, equivalently, relative to the counter top) would be the same as its speed relative to the lazy susan. However, as the cockroach is walking across the lazy susan, the lazy susan is rotating in the *opposite* direction, tending to take away from the distance covered by the roach relative to the counter top. The speed of the cockroach relative to the earth, v, is thus the speed of the roach relative to the lazy susan *minus* the speed of the rim of the lazy susan relative to the earth (this is the tricky part of this example – *think* about it and make sure that you feel comfortable with it — this *should* make sense to you!). Thus,

$$v = v_{ls} - V = 0.025\,\frac{m}{s}.$$

(h) What is the magnitude of the lazy susan's angular momentum while the cockroach is walking?

By definition, the angular momentum of the lazy susan is given by

$$L_{ls} = I_{ls}\,\omega = 1.0 \times 10^{-4}\,\frac{kg \cdot m^2}{s}.$$

(Do you see how we got the units for L? Remember that the *radian* is really a unitless unit – it can come and go as it pleases!)

(i) What is the magnitude of the cockroach's angular momentum?

Since the total angular momentum must be *zero*, it follows that

$$\mathbf{L}_{total} = \mathbf{L}_{ls} + \mathbf{L}_{cr} = 0 \qquad \text{or} \qquad \mathbf{L}_{cr} = -\mathbf{L}_{ls}.$$

The only way this can be true is of the magnitudes of the two angular momenta are equal (the minus sign just says that they are in opposite directions). Thus

$$L_{cr} = L_{ls} = 1.0 \times 10^{-4}\,\frac{kg \cdot m^2}{s}.$$

(j) What is the mass of the cockroach?

Combining our answers parts (b), (g) and (i), we get that (be careful here!)

$$L_{cr} = I_{cr}\,\omega_{cr} = \left(mR^2\right)\left(\frac{v}{R}\right) = 1.0\times10^{-4}\ \frac{kg\cdot m^2}{s}$$

$$\Rightarrow L_{cr} = mRv$$

$$\Rightarrow m = \frac{L_{cr}}{Rv} = 0.016\ kg = 16\ g\,.$$

Note that we used the fact that $v = \omega R$ to get the angular velocity of the cockroach. Although this example was stretched out into a million parts, it is nevertheless a typical conservation of angular momentum-type problem – things are initially constant (zero motion in this case), and then something internal happens to change the motions, but this happens in such a way that the total angular momentum of the system is *conserved.* You will, of course, have more opportunity to work with the conservation of angular momentum in the Problems Lab.

Kinetic Energy Revisited

We already know that a moving object has associated with it an *energy of motion*, called ***kinetic energy***. The kind of motion that we've spoken of until recently has been motion associated with the object physically moving through space – that is, its center-of-mass (CM) moves or *translates* through a displacement from one position to another. Such kinetic energy is thus called the ***translational kinetic energy***:

$$KE_{trans} = \tfrac{1}{2} Mv^2 .$$

We have now seen, however, that an object's *CM* need not move through space in order for there to be motion. Rather, the entire object can rotate about an axis through the object's *CM* such that the object does indeed have *motion* (angular motion) even though it isn't going anywhere. Associated with this motion is, of course, an *energy of motion*. This energy of motion is called ***rotational kinetic energy*** and is given by (*did you guess it?*)

$$KE_{rot} = \tfrac{1}{2} I\omega^2 .$$

The expression above for rotational kinetic energy must be used in any conservation of energy problems which involve an object rotating about its *CM*.

Rolling Motion

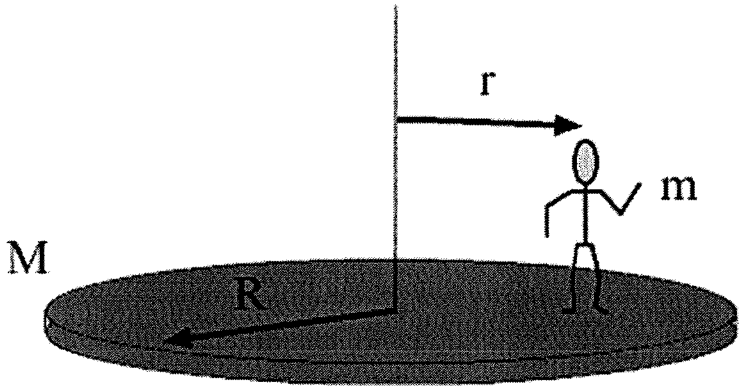
When an object *rolls*, it has *two* parts to its motion: a ***rotational part*** dealing with the spinning motion of the object about its axis of rotation, and a ***translational part*** dealing with the entire object moving as dictated by its axis of rotation moving through space. The total kinetic energy of a rolling object is therefore of the form (for an object rotating about an axis through its center of mass — the only case that we will consider...)

$$\begin{aligned} KE_{total} &= KE_{trans} + KE_{rot} \\ &= \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 \\ &= \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\left(\tfrac{v}{R}\right)^2 , \end{aligned}$$

since $v = \omega R$. When dealing with the conservation of energy, therefore, problems are approached the same as before, except that in the case of *rolling* motion, both the translational *and* the rotational kinetic energy must be taken into account. (The homework and the spreadsheets in the Problems Lab will provide you with opportunities to see how this works.) The next example will show how rotational kinetic energy can enter into work and power problems involving rotation.

Solution to Example 17.4

A *merry-go-round* with frictionless bearings consists of a solid circular platform of radius ***3.6 m*** and mass ***235 kg***. A friend of yours of mass ***78 kg*** is standing on the platform a distance of ***2.2 m*** from its center.



Data: R = 3.6 m M = 235 kg m = 78 kg r = 2.2 m

(a) What is the moment of inertia of the *platform* + *friend* system?

This system is composed of two parts: the merry-go-round ("*mgr*") and your friend ("*f*"). The moment of inertia of the system is then simply the moment of inertia of the merry-go-round plus the moment of inertia of your friend about the axis of rotation:

$$I = I_{mgr} + I_f .$$

The merry-go-round may be thought of as a solid disk of mass *M* and radius *R*. Its moment of inertia about its axis of symmetry (the axis through its center) is thus (from the table of Moments of Inertia)

$$\mathbf{I_{mgr} = \tfrac{1}{2} MR^2 = 1{,}500\ kg \cdot m^2} .$$

Also, since the size of your friend is small compared to the size of her distance from the axis of rotation, it won't be a bad approximation to treat her as a point mass *m* a distance *r* from the axis of rotation. Her moment of inertia is thus

$$\mathbf{I_f = mr^2 = 380\ kg \cdot m^2} .$$

It then follows that the system's rotational inertia is

$$I = I_{mgr} + I_f = 1{,}900 \text{ kg} \cdot \text{m}^2 .$$

(b) You push on the edge of the platform in the tangential direction to get the merry-go-round rotating. With what force should you push if the angular acceleration of the platform is to be ***2.1 rad/s²***?

The force you are exerting on the platform exerts a *torque* on the merry-go-round about the axis of rotation, resulting in the angular acceleration $\alpha = 2.1 \text{ rad/s}^2$. The torque that you exert is given by

$$\tau = r_{perp} F = R F .$$

But, since this is the *only* torque acting to rotate the system about its axis of rotation, Newton's 2nd law in rotational form gives us that

$$\sum \tau = I\alpha$$
$$\tau = RF = I\alpha$$
$$F = \frac{I\alpha}{R} = 1{,}100 \text{ N}$$

or about *250 lbs*.

(c) What is the angular speed of the platform after ***2.0 s*** of pushing if it started from rest?

This should obviously be a rotational kinematics problem:

$$\theta_i = 0 \quad \theta_f = ? \quad \omega_i = 0 \quad \omega_f = ? \quad \alpha = 2.1 \text{ rad/s}^2 \quad t = 2.0 \text{ s}$$

To solve for ω_f, we use the equation which is missing q_f:

$$\omega_f = \omega_i + \alpha t = \alpha t = 4.2 \text{ rad/s} .$$

(d) What is the period of rotation of the platform rotating at the speed in *(c)*?

By definition,

$$\omega_f = \frac{2\pi}{T} \rightarrow T = \frac{2\pi}{\omega_f} = 1.5 \text{ s}.$$

(e) What is the angular momentum of the system after the *2.0-s* interval?

Again, by definition,

$$L_f = I\omega_f = 7{,}980\,\frac{kg\cdot m^2}{s}.$$

(f) What is its kinetic energy at this time?

$$KE_{rot,f} = \tfrac{1}{2}I\omega_f^{\,2} = 16{,}800J.$$

(g) How much work did you do in pushing the platform for the *2.0-s* interval?

To answer this question, we will use the work-energy theorem. Remember that this theorem states that the net work done on an object is equal to the change in kinetic energy of the object. Thus, since the only force that does any work in rotating the platform is the force that you exert, the net work is also the work done by you. Then, since the system is initially not moving, we get that

$$W = \Delta(KE) = KE_{rot,f} - KE_{rot,i} = KE_{rot,f} = 16{,}800J.$$

Note that if the merry-go-round's bearing were *not* frictionless, then the (negative) work done by friction should also be included in the net work done on the system.

(h) What was your average power output during the *2.0-s* interval?

Remember that the average power is just energy or work divided by the corresponding time. Thus,

$$P = W/t = 8{,}400\ J.$$

The Vector Nature of Quantities

We have neither the need nor the time to get into a full discussion of the vector nature of the angular variables that we've defined so far. Nevertheless, it is advantageous to know at least some basic directions associated with a few of the angular quantities.

Consider an object is rotating about an axis. If you move the fingers of your *right* hand around the axis of rotation in the same direction that the object is rotating, then the extended thumb of your right hand will point along the axis in the direction of the **angular velocity** vector, $\boldsymbol{\omega}$.

If ω is *increasing* with time (so the object is spinning *faster*), then the **angular acceleration** $\boldsymbol{\alpha}$ points in the *same* direction as $\boldsymbol{\omega}$. (They point in *opposite* directions if the object is *slowing down*.)

The **angular momentum** vector $\boldsymbol{L}$ points in the same direction as $\boldsymbol{\omega}$ (since $\boldsymbol{L} = \boldsymbol{I\omega}$). *All of these angular quantities point one way or another along the axis of rotation.* (Note, however, that the torque does *not* follow this rule. It is very strange, and we will not go into it further here.)

These points will be addressed a little further in the Problems Lab. Ask your instructor if you are interested in learning more about the directions associated with these angular quantities and how they are related to one another – it's a fascinating subject, but very *weird*!

Summary Table 17

The following table summarizes the linear and angular quantities of interest along with their units. You should make sure that you understand how and when to apply these relations in solving problems.

Linear		Angular		Notes
x	linear displacement (m)	θ	angular displacement (rad)	$x = r\,\theta$
v	linear speed (m/s)	ω	angular speed (rad/s)	$v = r\,\omega$
a_t	linear (tangential) acceleration (m/s^2)	α	angular acceleration (rad/s^2)	$a_t = r\,\alpha$
$\sum \vec{F}$	net force (N)	$\sum \vec{\tau}$	net torque (N·m)	$\tau = r_{perp}\,F$
m	mass or translational inertia (kg)	I	rotational inertia or moment of inertia ($kg{\cdot}m^2$)	See the table of moments of inertia.
$\sum \vec{F} = m\vec{a}$	Newton's 2nd	$\sum \vec{\tau} = I\vec{\alpha}$	Newton's 2nd law for rotation	
$\vec{p} = m\vec{v}$	linear momentum (kg·m/s)	$\vec{L} = I\vec{\omega}$	angular momentum ($kg{\cdot}m^2/s$ or J·s)	
$\sum \vec{F} = \frac{\Delta \vec{p}}{\Delta t}$	Newton's 2nd law rewritten	$\sum \vec{\tau} = \frac{\Delta \vec{L}}{\Delta t}$	Newton's 2nd law rewritten	Shows that a net force or torque changes an object's momentum
	conservation of linear		conservation of angular	

$\Sigma \vec{p}_i = \Sigma \vec{p}_f$	momentum (if $\mathbf{F}_{net} = 0$)	$\Sigma \vec{L}_i = \Sigma \vec{L}_f$	momentum (if $\tau_{net} = 0$)	
$KE_{trans} = \frac{1}{2} M v^2$	translational kinetic energy (J)	$KE_{rot} = \frac{1}{2} I \omega^2$	rotational kinetic energy (J)	Remember that $v = \omega r$ for rolling motion.

Sample Quiz 17

Circle the letter of the best answer.

A solid sphere ***($I = 2/5\ MR^2$)*** of radius ***12 cm*** is mounted on a vertical frictionless axis about which it is free to rotate (kind-of like a globe). You apply a force of ***120 N*** tangentially to the equator of the sphere. The sphere is found to have a resulting angular acceleration of ***5.8 rad/s²***.

1. What is the torque exerted by the force on the sphere about its axis?

a. 7.2 N m b. 14 N m c. 28 N m d. 45 N m e. 72 N m

2. What is the moment of inertia of the sphere? *Hint: The equation given above will not help in this part!*

a. 2.4 kg m^2 b. 5.7 kg m^2 c. 8.2 kg m^2 d. 12 kg m^2 e. 21 kg m^2

3. What is the mass of the sphere?

a. 57 kg b. 85 kg c. 130 kg d. 360 kg e. 420 kg

4. What is the rotational kinetic energy of the sphere if it is rotating with an angular speed of ***2.7 rad/s***?

a. 2.1 J b. 3.7 J c. 6.3 J d. 8.8 J e. 9.5 J

5. What are the *MKS* units of *angular momentum*?

a. kg b. kg m/s c. kg m^2 d. kg m^2/s e. kg m^2/s^2

Answers

Answers to Sample Quiz 17

1. b

2. a (Hint: $\tau = I\, a$!)

3. e

4. d

5. d

Homework 17

I. Warm-up Exercises

1. A hollow sphere of mass **4.5 kg** and diameter **64 cm** is rotated about an axis through its center. What is the moment of inertia of this sphere?

2. A solid sphere of mass **4.5 kg** and diameter **64 cm** is rotated about an axis through its center. What is the moment of inertia of this sphere?

3. The sphere in problem #2 above is rotating with an angular velocity of **3.4 rad/s**. (a) What is its angular momentum magnitude? (b) What is its kinetic energy?

4. Explain as clearly and concisely as possible the concepts of translational and rotational inertia. Give the MKS units of each of these quantities.

5. A courageous student stands on a quasi-frictionless platform while holding two masses of considerable weight in his hands. With his arms extended horizontally outward, a fellow student gets him spinning slowly on the platform. The spinning student then starts pulling his arms inward. As he does so, he starts spinning faster and faster. When he once again extends his arms, he slows down to his original rotational speed. Explain what is going on.

6. The most widely accepted theory of stellar formation contends that a star forms from the collapse of a huge cloud of gas (usually composed primarily of hydrogen) in interstellar space. The cloud's collapse is due to the gravitational pull that each part of the cloud exerts on each of the other parts. As the cloud of gas collapses, internal friction causes the temperature of the gas to increase, which in turn causes the pressure to increase. Eventually, if the cloud has sufficient mass (and thus sufficient gravitational pull), the temperature gets high enough so that nuclear reactions can start within its core. A star is then born. One thing that most people don't know is that all stars have some amount of rotation about their axes – in fact, some types of very compact stars can rotate more than 100 times per second! This very quick rotation results from the initial rotation of the cloud from which the star formed, which is always an extremely slow residual rotation.

Explain as clearly and completely as possible the last sentence in the paragraph above.

II. Some Standards

7. A spinning, hollow ball of mass **M = 0.35 kg** and radius **R = 12 cm** is

found initially to rotate at a uniform rate of **4 rotations** every second. (a) What is the angular velocity of the ball? (b) What is its moment of inertia? (c) What is its angular momentum? (d) The ball then starts shrinking in size (for some unknown and unimportant reason...). How many times per second is it spinning when its radius is **6.0 cm**?

8. A solid sphere of mass **3.5 kg** and radius **5.2 cm** is placed at the top of an incline of height **12.0 cm** and length **3.5 m**. The sphere is released from rest and rolls down the incline without slipping. How fast is the sphere moving (that is, what is the speed of its center of mass) when it reaches the bottom of the incline?

9. A solid cylinder of mass **3.5 kg**, radius **5.2 cm** and length **7.4 cm** is placed at the top of an incline of height **12.0 cm** and length **3.5 m**. The cylinder is released from rest and rolls down the incline without slipping. How fast is the cylinder moving (that is, what is the speed of its center of mass) when it reaches the bottom of the incline?

III. So, you think you're pretty good...?

10. A cockroach of mass **1.7 g** runs counterclockwise around the rim of a lazy susan of radius **7.2 cm** and moment of inertia **7.7×10^{-5} kg m^2** having frictionless bearings. The cockroach's speed *relative to the earth* is **0.35 cm/s**, whereas the lazy susan turns clockwise (taken to be the positive direction for rotation) with an angular speed of **0.033 rad/s**. The cockroach suddenly finds a bread crumb on the rim and, of course, stops running. (a) What is the initial angular momentum of the cockroach? (Be careful of the sign!) (b) What is the initial angular momentum of the lazy susan? (c) What is the total initial angular momentum of the system? (d) What is the total angular momentum of the system after the cockroach stops? (e) What is the total moment of inertia of the system after the cockroach stops? (f) What is the angular speed of the lazy susan (and cockroach) after the cockroach stops running?

Answers

Answers to Homework 17

1. 0.31 kg m^2

2. 0.18 kg m^2

3. (a) 0.61 kg m^2/s (b) 1.0 J

4. *See notes.*

5. Conservation of angular momentum: as I decreases, ω increases to keep L a constant.

6. Conservation of angular momentum

7. (a) 25 rad/s (b) 0.0034 kg m^2 (c) 0.086 kg m^2/s (d) 16 (*Hint:* Find I_f, and then use the conservation of angular momentum to find ω_f.)

8. 1.30 m/s (*Hint:* conservation of energy! If you use only symbols until the very end, you will find that the only information that you need to know about the sphere is that it is solid, and that it's released from a height of 12 cm. You do *not* need the mass or radius of the sphere to solve this problem!)

9. 1.25 m/s (Same comments as above.)

10. (a) – 4.28 x 10^{-7} kg m^2/s (b) 2.54 x 10^{-6} kg m^2/s (c) 2.11 x 10^{-6} kg m^2/s (d) 2.11 x 10^{-6} kg m^2/s (e) 8.6 x 10^{-5} kg m^2 (f) 0.025 rad/s (conservation of angular momentum!)